BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

EDITED BY

F. N. COLE VIRGIL SNYDER J. W. YOUNG

ALEXANDER ZIWET D. E. SMITH

T. LEVI-CIVITA R. C. ARCHIBALD

VOL. XXII

OCTOBER 1915 TO JULY 1916

PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK
1916

Press of
The New Era Printing Company
Lancaster, Pa

BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

THE TWENTY-SECOND SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The twenty-second summer meeting of the Society was held at the University of California on Tuesday, and at Stanford University on Wednesday, August 3–4, 1915, in connection with the Panama-Pacific International Exposition. The following thirty-four members of the Society were in attendance upon the two sessions:

Professor R. E. Allardice, Dr. Nathan Altshiller, Dr. Charlotte C. Barnum, Dr. A. A. Bennett, Dr. B. A. Bernstein, Professor H. F. Blichfeldt, Professor C. E. Brooks, Dr. Thomas Buck, Professor E. F. A. Carey, Professor E. E. De Cou, Dr. H. B. Curtis, Professor G. C. Edwards, Dr. Elizabeth B. Grennan, Professor F. L. Griffin, Professor F. W. Hanawalt, Professor Charles Haseman, Professor M. W. Haskell, Professor E. R. Hedrick, Professor L. M. Hoskins, Dr. Frank Irwin, Professor C. J. Keyser, Professor D. N. Lehmer, Professor J. H. McDonald, Professor W. A. Manning, Professor H. C. Moreno, Professor R. E. Moritz, Professor L. I. Neikirk, Professor C. A. Noble, Professor E. W. Ponzer, Mr. F. D. Posey, Professor T. M. Putnam, Professor H. W. Stager, Professor H. W. Tyler, Professor S. E. Urner.

Professor M. W. Haskell, chairman of the San Francisco Section, presided at the session on Tuesday afternoon, and Professor R. E. Allardice at that on Wednesday afternoon.

Tuesday morning was devoted to a joint session with the American Astronomical Society and Section A of the American Association for the Advancement of Science. Addresses were delivered by Professors C. J. Keyser on "The human significance of mathematics," and G. E. Hale on "The work of a modern observatory." The astronomers, mathematicians, and physicists lunched at the Faculty Club as guests of Professors Leuschner, Haskell, and E. P. Lewis.

The social programme included a dinner with the American Astronomical Society at the Hotel Oakland on Wednesday evening, an excursion to the Lick Observatory on Friday, and a luncheon given by Mrs. Phoebe Hearst at the Hacienda del Pozo de Verona on Saturday.

The following papers were presented at this meeting:

(1) Professor L. E. DICKSON: "Invariantive classification of pairs of conics modulo 2."

(2) Professor C. J. DE LA VALLÉE POUSSIN: "Sur l'intégrale de Lebesgue."

(3) Mr. A. R. SCHWEITZER: "On the solution of a class of functional equations."

(4) Dr. NATHAN ALTSHILLER: "On the circles of Apollonius."

(5) Dr. DUNHAM JACKSON: "Proof of a theorem of Haskins."

(6) Dr. W. W. KÜSTERMANN: "Fourier constants of functions of two variables."

(7) Dr. B. A. BERNSTEIN: "A set of four independent postulates for Boolean algebras."

(8) Professor G. A. MILLER: "Limits of the degree of transitivity of substitution groups."

(9) Professor R. D. CARMICHAEL: "On the representation of numbers in the form $x^3 + y^3 + z^3 - 3xyz$."

(10) Professor H. S. WHITE: "Seven points on a gauche cubic curve."

(11) Professor M. W. HASKELL: "The del Pezzo quintic curve."

(12) Professor L. J. RICHARDSON: "A phase of Roman mathematics."

(13) Dr. C. A. FISCHER: "Functions of surfaces with exceptional points or curves."

(14) Mr. A. R. WILLIAMS: "On a birational transformation connected with a pencil of cubics."

(15) Professor F. N. COLE: "Note on the triad systems in 15 letters."

(16) Professor A. B. COBLE: "The determination of the lines on a cubic surface."

(17) Mr. H. S. VANDIVER: "An aspect of the linear congruence, with applications to the theory of Fermat's quotient."

(18) Dr. C. H. FORSYTH: "An interpolation formula based upon central and multiple differences."

(19) Dr. G. M. Green: "On isothermally conjugate nets of space curves."

(20) Professor L. P. Eisenhart: "Surfaces of rolling and transformations of Ribaucour."

(21) Mr. A. R. Schweitzer: "Generalized quasi-transitive functional relations."

(22) Professor L. M. Hoskins: "'Quantity of matter' in dynamics."

(23) Dr. A. A. Bennett: "The iteration of functions of one variable."

Professor Richardson was introduced by Professor Haskell and Mr. Williams by Professor Lehmer. Professor de la Vallée Poussin's paper was communicated to the Society through Professor Birkhoff. Professor White's paper was read by Professor Haskell. The papers by Professor Dickson, Professor de la Vallée Poussin, Mr. Schweitzer, Dr. Jackson, Dr. Küstermann, Professor Miller, Professor Carmichael, Dr. Fischer, Professor Cole, Professor Coble, Mr. Vandiver, Dr. Forsyth, Dr. Green, and Professor Eisenhart were read by title.

Abstracts of the papers, including Professor Keyser's address, follow below. The abstracts are numbered to correspond to the papers in the list above.

In Professor Keyser's address the treatment, conducted in the spirit appropriate to an international exposition, aims at being interesting and intelligible to the general educated public. A sketch of the manner in which a historian of mathematics might hope to vindicate the science is followed by a more elaborate exposition of the task of attaining the same end through an account of the present state of the science. The claims of mathematics to human regard as based on its applications in a wide range of other sciences and arts are next dealt with. This matter is followed by a characterization of the modern critical movement of the science and of the resulting conception of the distinctive character of mathematics, especially in its relation to modern developments in logic. The chief emphasis of the address falls on the bearings of the science as distinguished from its applications—upon the significance of mathematics for man regarded as finding his supreme interest in seeking permanent values in an inpermanent world.

1. With a conic F modulo 2 is associated covariantively a point A, called its apex, and a unique line L, and conversely A and L uniquely determine F (Madison Colloquium Lectures, 1914, page 69). Hence the projective classification of pairs of conics F and F' is equivalent to that of the systems A, L, A', L' of two points and two lines and the degenerate systems in which one or more of the four elements are absent. A simple geometrical discussion of such systems leads Professor Dickson to the theorem: Two pairs of conics modulo 2 are projectively equivalent if and only if they have the same properties as regards existence of apices and covariant lines, distinctness of apices and lines, and incidence of apices and lines. These properties are expressed analytically by very simple modular invariants, which therefore form a fundamental system of modular invariants of two conics.

2. Professor de la Vallée Poussin's paper, which reproduces a part of his recent course of lectures at Harvard University, will appear in the *Transactions*.

3. The object of Mr. Schweitzer's paper is to discuss the solution of the equations

$$f\{\lambda_1 f(t_1, t_2, \cdots, t_n, x_1), \lambda_2 f(t_1, t_2, \cdots, t_n, x_2), \cdots, \\ \lambda_{n+1} f(t_1, t_2, \cdots, t_n, x_{n+1})\} = \mu f\{x_{i_1}, x_{i_2}, \cdots, x_{i_{n+1}}\},$$

where $n = 1, 2, 3$, etc.; the subscripts $i_1, i_2, \cdots, i_{n+1}$ are distinct and range respectively over the values $1, 2, \cdots, n + 1$; λ_i and μ denote functions of a single variable. The $(n + 1)!$ functional equations thus defined may be put into (1, 1) relation with the substitutions on $n + 1$ symbols and therefore conveniently denoted by the notation

$$E\left\{\begin{pmatrix} 1 & 2 & 3 & \cdots & (n+1) \\ i_1 & i_2 & i_3 & \cdots & i_{n+1} \end{pmatrix}; \lambda_1(x), \lambda_2(x), \cdots, \lambda_{n+1}(x), \mu(x)\right\}.$$

For $n = 1$ the equations have been completely discussed by the author in previous articles. For $n > 1$ the special instance of identity of some or all of the functions $\lambda_i(x)$, $\mu(x)$ is considered. In the simplest case, namely, when $n = 2$ and the functions $\lambda_i(x)$, $\mu(x)$ are all identical and equal to x, the following theorems on the equations

$$E\left\{\begin{pmatrix} 1 & 2 & 3 & \cdots & (n+1) \\ i_1 & i_2 & i_3 & \cdots & i_{n+1} \end{pmatrix}\right\}$$

are valid:

1. $E\{(1)\ (2)\ (3)\}$ implies the existence of $\psi(x)$ such that $f(x_1, x_2, x_3) = \psi^{-1}\{[\psi(x_3) - \psi(x_2)] + \theta[\psi(x_2) - \psi(x_1)]\}$, where $\theta(x)$ is arbitrary.
2. $E\{(1)\ (23)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{- 2\psi(x_1) + \psi(x_2) + \psi(x_3)\}$.
3. $E\{(2)\ (13)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{\psi(x_1) - 2\psi(x_2) + \psi(x_3)\}$.
4. $E\{(3)\ (12)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{[\psi(x_3) - \psi(x_2)] + \theta[\psi(x_2) - \psi(x_1)]\}$, $\theta(x) = x + \theta(- x)$.
5. $E\{(123)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{\psi(x_1) + c\psi(x_2) + c^2\psi(x_3)\}$, where $1 + c + c^2 = 0$.
6. $E\{(132)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{c\psi(x_1) + \psi(x_2) + c^2\psi(x_3)\}$, where $1 + c + c^2 = 0$.

Solutions 2–4 are special cases of the Solution 1.

4. In his contribution to the modern geometry of the triangle Dr. Altshiller, starting with the usual definition of the circles of Apollonius connected with a triangle, derives, in a purely synthetic way, a series of properties, partly known, involving the Lemoine point, the symmedian lines, the Brocard diameter, etc., and leads up to the theorem: The center of any one Apollonian circle is a center of similitude of the two other circles of Apollonius, the second center of similitude being the pole of the Brocard diameter with respect to the first circle. This paper will appear in the *American Mathematical Monthly.*

5. It is a theorem due to Haskins that if f and φ are two bounded functions such that the definite integral of f^n over an interval is equal to that of φ^n for all positive integral values of n, the set of points where $\alpha \leqq f \leqq \beta$ and the set where $\alpha \leqq \varphi \leqq \beta$ have the same measure, for any pair of numbers α, β. Dr. Jackson gives a short proof of this theorem, based on a polynomial approximation and Lebesgue's theorem on the integration of a uniformly bounded sequence. The result is of particular interest in the case where the functions are monotone.

6. In volume 57 of the *Mathematische Annalen* A. Hurwitz has shown how to express the product of two ordinary Fourier

series in form of another Fourier series, or, stated differently, how to compute the Fourier constants of the product of two functions from those of the functions themselves. Dr. Küstermann solves the analogous problem for double Fourier series. The solution depends vitally upon the proof of the relation

$$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[f(x, y)]^2dxdy = \sum_{\mu,\,\nu=0}^{\infty}\frac{(a^2+b^2+c^2+d^2)_{\mu\nu}}{2^{E[1/(\mu+1)]+E[1/(\nu+1)]}},$$

where $(a, b, c, d)_{\mu\nu}$ are the Fourier constants of $f(x, y)$ presumed to be bounded and integrable in the Riemann sense. The analogue of this relation for functions of a single variable is due to Parseval and was proved by him under certain restrictions on the nature of convergence of the Fourier series involved. In 1893 de la Vallée Poussin gave a proof requiring merely that the function and its square be integrable. Hurwitz, in 1903, again called attention to the wide scope of the theorem, gave a new proof, and applied it to the problem mentioned. More recently the formula has gained in general interest through the researches of Riesz and Fischer (Riesz-Fischer theorem). In the present paper the proof is established by means of Poisson's double integral.

7. The most economical set of postulates for the logic of classes is that presented by H. M. Sheffer in the *Transactions* of October, 1913. Using as basis the operation of "rejection," which Dr. Sheffer used, Dr. Bernstein obtains a set of four postulates from which the former's set of five can be deduced. The paper will appear in the *Transactions*.

8. The main theorem established by Professor Miller in the present paper is as follows: A substitution group of degree $n = kp + r$, where p is a prime number and r and k are positive integers such that $p > k$ and $r > k$, cannot be more than r times transitive unless either this group includes the alternating group of degree n or $k = 1$ and $r = 2$. By means of this theorem it can easily be proved that a group which does not include the alternating group of its degree cannot be as much as 15-fold transitive unless its degree exceeds 1,000. It is also proved that whenever $n > 12$ the theorem will always give a smaller upper limit for the degree of transitivity of a substitution group which does not include the

alternating group of its degree than the limit obtained by using the well-known formula $\frac{1}{3}n + 1$. The present paper is an extension of results obtained by the same writer in an article published in the BULLETIN, volume 4 (1898), page 140, and it aims to replace the formula $\frac{1}{3}n + 1$ for the upper limit of the degree of transitivity of the groups in question by a theorem which gives at least as low limits when $n > 7$, and gives always a lower limit when $n > 12$.

9. Among the theorems presented by Professor Carmichael in this note are the following: Every prime number other than 3 can be represented in one way and in only one way in the form $x^3 + y^3 + z^3 - 3xyz$, where x, y, z are non-negative integers; primes of the form $6n + 1$ have one additional representation in which at least one of the integers x, y, z is negative.

10. In previous theories seven points on a twisted cubic curve have been studied only in combinations that were symmetric; and the only theorem known, beyond what may be considered as definition of the curve, states that the 35 points where the seven osculating planes meet must lie on a surface of order five. Professor White examines sets of seven derived points corresponding to a triad system on seven elements, and finds them to lie upon a second cubic curve, one of thirty determined by the first seven points. Analytic proofs are given.

11. Professor Haskell's paper contains the following items: Reduction of the Hessian of the plane quintic with five cusps, showing that the five cusps and five inflexions lie on a cubic, and that the inflexions are rational in terms of the cusps. Determination of a conic with regard to which the quintic is self-dual, of a Cremona transformation of the quintic into itself, and of sets of conics and cubics connected with the cusps and inflexions.

12. Professor Richardson's paper gives a practical illustration of manual multiplication as practised by the Romans, together with a brief historical survey of this system of reckoning.

13. In a paper published in the July, 1914, number of the *American Journal of Mathematics,* Dr. Fischer has given a

definition of the derivative of a function of a surface analogous to Volterra's definition of the derivative of a function of a line, and has proved that if the derivative is continuous and approached uniformly, the first variation of the function is equal to the double integral of the derivative multiplied by the first variation of the dependent variable. In the present paper this theory is extended to the case where there are points or curves where the derivative does not exist. The theory of exceptional points is used in finding the second variation of a function of a double integral, and the theory of exceptional curves is applied to the variable boundary problem in the calculus of variations.

14. The theorem that all the cubics through eight points pass through a ninth, suggests the following transformation which is evidently birational: Fix seven of the eight points, and make correspond to a variable eighth the ninth point through which all the cubics pass.

Dr. Hart (see *Cambridge and Dublin Mathematical Journal* volume 6, page 181) has given a synthetic construction for the ninth point determined by eight given points. By an analytic treatment of this construction Mr. Williams has found the following relation between the coordinates, x', y', z', of 9 and the coordinates of 8: $x' = C_1C_2K_1$; $y' = C_2C_3K_2$; $z' = C_3C_1K_3$, where the C's are cubic functions and the K's are quadratic. These equations define a Cremona transformation.

In the above transformation there are infinitely many invariant points, all of which lie on a sextic curve of deficiency three, having double points at each of the seven fixed points. At each of these double points the tangents to the sextic coincide with the tangents to the nodal cubic having the same double point and passing through the other six fixed points. The sextic also meets the line determined by any two of these seven points in the points where this line meets the conic determined by the other five. From this follows the remarkable theorem concerning seven arbitrary points in the plane: if the line joining any two of them be cut by the conic determined by the other five, the forty-two points thus obtained lie on a sextic. This sextic satisfies seventy-seven conditions, almost three times the number necessary to determine it.

By introducing relations among the seven given points there result other types of Cremona transformations with their corresponding invariant curves. Thus if three of the seven points are on a line, we have a transformation of the seventh degree with an invariant quintic curve. And this quintic satisfies fifty-three conditions. Similarly, if six of the seven points lie on a conic, there results a quartic transformation with an invariant quartic curve fulfilling forty-seven conditions. If six points are on a conic and at the same time three are on a line, a cubic transformation results with an invariant cubic curve fulfilling twenty-nine conditions. If six points are on a conic and if at the same time three are on one straight line and three on another, the transformation is quadratic, and the invariant curve is a conic which fulfills sixteen conditions. This quadratic transformation is of a type of which inversion with respect to the unit circle is a special case.

15. Professor Cole has found, by detailed examination of the various cases, that every triad system in 15 letters, with a single exception, presents an interlacing, $(xab)(xcd)$, $(yac)(ybd)$, the exceptional case being one of Heffter's systems. It is also found that every triad system in 15 letters presents either a hexad, $(xab)(xcd)(xef)$, $(yac)(yde)(ybf)$, or triple tetrads, $(xab)(xcd)(xef)(xgh)(xij)(xkl)$, $(yac)(ybd)(yeg)(yfh)(yik)(yjl)$.

16. In earlier reports Professor Coble has discussed the Cremona group $G_{6,2}$ in Σ_4 of order 51840 determined by a P_6^2 (six points in a plane). It is the purpose of this paper to show that $G_{6,2}$ furnishes the direct algebraic connection between the cubic surface C^3, mapped on the plane by cubic curves on P_6^2, and the collineation groups which arise in the trisection of the theta functions of genus 2. This connection is made by the use of irrational invariants of C^3 which are associated with the separation of the lines of the surface. The discriminant of C^3 consists of 36 irrational factors corresponding to the 36 double sixes of C^3. By forming products of 9 such factors properly selected, a set of 40 irrational "complex invariants" is obtained. These determine in Σ_4 a set of 40 quintic spreads conjugate under $G_{6,2}$. The spreads however lie in a linear system of dimension 9 and one can

choose a linearly independent set, $X_0, \cdots, X_4$; $U_0, \cdots, U_4$, with the striking property developed below.

On the other hand Klein had discovered and Burkhardt had developed a collineation group in S_4 with variables $Y_0, \cdots, Y_4$ of order 25920. If to this group there be added the correlations which transform it into itself there is obtained a mixed group g_{51840} in the contragredient variables $Y_0, \cdots, Y_4$; $V_0, \cdots, V_4$, which is isomorphic with $G_{6,2}$ and which has the following property: The variables Y, V are transformed under g_{51840} precisely as the irrational invariants X, U above are transformed under $G_{6,2}$.

This property leads to the rational determination of the lines of a cubic surface (after the adjunction of the square root of its discriminant) in terms of the solution of the form problem of Burkhardt's group without passing beyond the domain of the irrational invariants of the given surface. The methods hitherto suggested for this purpose required the introduction of an equation of degree 27 (or a resolvent of some other degree), i. e., the introduction of a binary field necessarily extraneous to the surface itself.

17. In 1903, Professor G. D. Birkhoff communicated to Mr. Vandiver the following theorem:

If p is a prime integer and a is a positive integer prime to p, then there is at least one and not more than two sets (x, y) such that

$$a \equiv \pm x/y \pmod{p},$$

where x and y are integers prime to each other and $0 < x < \sqrt{p}$, $0 < y < \sqrt{p}$.

In the present paper a proof of this theorem is given, involving a continued fraction algorithm for the direct determination of each set. It is also shown that the fact that there exists at least one set (x, y) follows from a theorem due to Minkowski. The least positive residues of the integers

$$1^{p-1}, \quad 2^{p-1}, \cdots, \quad (p-1)^{p-1}$$

modulo p^2 are termed *proper residues* modulo p^2. By an extension of the result mentioned above the following theorem is derived:

There are not more than

$$p - \tfrac{1}{2}(1 + \sqrt{2p-5})$$

and not less than $[\sqrt{p}]$ incongruent proper residues modulo p^2, if p is a prime > 2.

Applications to problems in connection with Fermat's last theorem are then discussed.

18. The accuracy of an interpolation based upon finite differences is increased as the values of the independent variable occurring in the differences approach the value of the independent variable of the ordinate which is interpolated. Dr. Forsyth has derived an interpolation formula wherein these values are identical.

Furthermore, instead of the usual averaging of differences (i. e., use of first differences) to supply differences which are missing (as always happens in central differences), differences are used whose orders are in keeping with the order of the interpolation formula itself.

19. In Dr. Green's note is given a new geometric interpretation of Bianchi's condition for isothermally conjugate nets of curves on a surface. It will appear in the *Proceedings of the National Academy of Sciences.*

20. In a note published in volume 23 of the *Rendiconti dei Lincei* Bianchi has defined as a surface of rolling the surface described by a point invariably fixed with respect to a surface S_0 as the latter rolls over an applicable surface S, which Bianchi calls the surface of support. He shows that, given any surface Σ, the problem of finding pairs of applicable surfaces S_0 and S such that Σ is a surface of rolling as S_0 rolls over S reduces to the integration of a partial differential equation of the second order and of a Riccati equation. Two surfaces are said to be in the relation of a transformation of Ribaucour when they constitute the envelope of a two-parameter family of spheres such that the lines of curvature correspond on the two surfaces, corresponding points being on the same sphere. Professor Eisenhart has shown that the necessary and sufficient condition that either surface of two so related be a surface of rolling with the locus of the centers of the spheres for surface of support is that the two surfaces be isothermic and in the relation of a so-called transformation D_m, discovered by Darboux.

21. As a quasi-transitive functional relation in the most general sense, Mr. Schweitzer defines

$$\phi\{f_1(t_1, t_2, \cdots, t_n, x_1), f_2(t_1, t_2, \cdots, t_n, x_2), \cdots, \\ f_{n+1}(t_1, t_2, \cdots, t_n, x_{n+1})\} = \psi(x_1, x_2, \cdots, x_{n+1}),$$

where $n = 1, 2, 3$, etc. From this class of equations other classes are deduced, first, by the homologous transposition of the x's in the left hand member and, second, by substituting x's for some or all of the t's in the set of equations thus obtained (including the original class). For example, one obtains by homologous transposition of the x's when $n = 2$

$$\phi\{f_1(x_1, t_1, t_2), f_2(x_2, t_1, t_2), f_3(x_3, t_1, t_2)\} = \psi(x_1, x_2, x_3)$$
$$\phi\{f_1(t_1, x_1, t_2), f_2(t_1, x_2, t_2), f_3(t_1, x_3, t_2)\} = \psi(x_1, x_2, x_3).$$

Interesting problems are presented by systems of equations belonging to the same or different classes.

22. The proposition that the accelerations of different bodies under the action of equal forces are inversely proportional to their masses is often asserted to be merely a definition of mass. The object of the paper by Professor Hoskins is to show that, in applying this proposition, our interpretation of it involves the notion of mass as a quantitative measure of the matter of which the bodies are composed.

23. Dr. Bennett considers several topics connected with the iteration of functions of one variable. Matrices with an infinite number of elements are used to obtain a classification of the types of series which are to be distinguished with respect to the nature of their iteration. The different types are considered, with particular reference to questions of convergence. The iteration of functions of a real variable is also considered. The paper will appear in the *Annals of Mathematics*.

THOMAS BUCK,
Acting Secretary.

GROUPLESS TRIAD SYSTEMS ON FIFTEEN ELEMENTS.

BY DR. LOUISE D. CUMMINGS AND PROFESSOR H. S. WHITE.

(Read before the American Mathematical Society April 24, 1915.)

FROM previous publications and a paper presented to this Society in October, 1914, 44 different triad systems on 15 elements (Δ_{15}) are known. These 44 systems separate into two types 23: of the systems each contain one or more systems

on 7 elements and have been designated as headed systems, the remaining 21 contain no Δ_7 and have been designated as headless systems. The groups for the 44 systems have been obtained and range in order from $8!/2$ down to 2. Hence for each of these systems there exists at least one substitution different from identity which transforms the system into itself.

Systems whose group is identity or *groupless* systems on 15 elements have not been known hitherto. No groupless systems exist for 7, 9, or 13 elements. In a paper published in the *Transactions*, January, 1915, one of us has proved the existence of millions of triad systems on 31 elements with the group identity. This raised the question concerning the existence of the groupless systems for 15 elements. About six months ago we began a rigorous search for this type of Δ_{15}, and, while the investigation is at present far from completion, we have had the good fortune to discover systems of this new species.

In the *Transactions* for January, 1913, Professor Cole discussed triad systems on 13 elements, and pointed out the fact that two elements in a system may be connected by what he has termed an "interlacing"; for example the four triads $a12$, $a34$, $b13$, $b24$ occurring in a system form an interlacing of the elements a and b. Interlacing and some extensions of this idea have proved to be of fundamental importance in the formation of the groupless systems.

All interlacings and the more extended forms of interconnection of the elements, which we make use of, may be obtained by applying to the system under consideration what may be called a contracted form of sequences and indices. The Δ_{15} is written in a 15 by 7 array. Each element heads one column; below it are placed the 7 pairs of elements which with the element at the head occur in the triads of the system. Heretofore, sequences and indices have been formed from the 3 columns of a triad in any Δ_{15}. The same process is now applied to every pair of columns of a Δ_{15}. Since the number of combinations of 15 columns, two at a time, is 105, 105 pairs of columns must be examined, unless the group for the system is already known and is different from identity. If the group contains an operator of order k, then k pairs of columns are examined simultaneously. The process may be illustrated in its application to a system J, with a group of order 4 generated by $s = (a)(bc)(d8e7)\ (f3g4)(1526)$. Pairs of

columns selected from the following table show every type of contracted index that occurs in any system.

a	b	d	1
de	df	ae	$a2$
fg	eg	bf	$b3$
12	ac	$c2$	ce
34	13	$g8$	$d5$
56	24	15	$f8$
78	57	37	$g4$
bc	68	46	67

Pair of Columns.	Index.	Contracted Sequences.
ab	2^3	$de/gf/d$, 12/43/1, 56/87/5;
$b1$	3^2	$df/86/75/d$, $24/ge/ca/2$;
da	6	$c2/15/64/37/8g/fb/c$;
db	2, 4	$ae/g8/64/2c/a$, 15/73/1.

The substitution s applied to the pair of columns ab gives the pair ac with the same index and similar sequences. If s is applied to the pair of columns $b1$, the three pairs $c5$, $b2$, $c6$ are obtained with the index 3^2. The analysis of the 105 pairs of columns shows that the contracted indices 2^3; 2, 4; 3^2; 6 belong, respectively, to 2, 24, 4, and 75 pairs of the columns.

The new groupless systems are formed by interchanging duads of one column with those of another column. For example, in the pair of columns ab, the duads de, fg of column a may be interchanged with df, eg of column b; such an interchange involving four elements, contained only in two pairs in each column, shall be designated as a *quadrangular transformation.* The columns d, e, f, g must now be re-written, in agreement with the new triads introduced into the system, and the undisturbed 9 columns of J with the re-constructed 6 columns form a new system JQ. The four duads 12, 34, 56, 78 of column a might be interchanged with the four duads on the same elements in column b, forming an octagonal transformation, but this is equivalent to the above quadrangular transformation followed by the interchange of the elements a and b.

In the pair of columns $b1$ the duads df, 68, 57 of column b may be interchanged with the duads $f8$, 67, $d5$ of column 1; such an interchange, involving six elements appearing exclusively in 3 pairs in each of the two columns, shall be designated as a *hexagonal transformation.* New columns d, f, 6, 8, 5, 7

must next be constructed; these eight reconstructed columns, with the seven undisturbed columns of J, form a system JH. The application of the second hexagonal transformation in $b1$ is equivalent to an application of the first hexagonal transformation followed by the interchange of the elements b and 1.

A transformation on 12 elements simply interchanges the two elements which head the columns.

Therefore only the quadrangular and the hexagonal transformations which exist in a system require consideration.

By means of the operators of the group of the system, the minimum number of non-congruent transformations of each of the above types is determined—for example, in J, the 8 hexagonal transformations reduce to 1, and the 30 quadrangular transformations to 4 non-congruent transformations.

Each of the non-congruent transformations is now applied to the system J, and the sequences and indices are determined for the five transformed systems.

The 35 triads of the system JH arranged in classes according to their indices are shown in the following table.

1^426	1^435	1^32^25	1^39	1^22^34	1^228	1^237	1^246	1^25^2
157	*a*12 258	*a*78 *a*56 *cg*5	168	*a*34	*g*14 *c*47 *dg*8	*cd*2	*b*24	*b*13
12^27	1236	1245	1 11	2^24^2	23^24	5, 7	6^2	
ade *d*46	*afg* *g*27	*e*26 *f*45 *cf*6 *c*38 *d*37	*bd*5 *bf*8 *b*67 *beg* *ce*1	*e*48 *df*1 *g*36	*ef*7	*e*35 *f*23	*abc*	

The enumeration of the entrances of the elements in the 17 classes shows that the transitive sets of elements are a; b; c; d; e; f; g; 1; 2; 3; 4; 5; 6; 7; 8; hence the group for the system is identity. Therefore under the hexagonal transformation the system J, with a group of order 4, is changed into a system JH with the group identity. The four quadrangular transformations applied to J yield four non-congruent systems. One of these is a new groupless system JQ; the remaining three are known headless systems with groups of orders 2, 2, and 6 respectively.

The work of transforming the 44 known systems has been partially performed and we have discovered about 20 systems of the new groupless type. The headed type of Δ_{15} is exhausted in 23 systems; the non-headed with a group, in 21. The numerous repetitions now appearing in the work lead us to believe that the groupless type is not much more numerous than the other two types. The purpose of the present investigation is to obtain, if possible, a complete set of groupless systems, and then to show the connecting links between all the systems of the three types.

The system JH with no head and no group, which has been derived by a hexagonal transformation from a system J, with no head, but with a group of order 4, may also be derived from another system IC with a head and a group of order 3. Hexagonal transformations leave unchanged the number of Δ_7's in a Δ_{15}, and, therefore, always transform systems with or without heads into systems with or without heads respectively. A quadrangular transformation may change the number of Δ_7's in the Δ_{15}. The indices of the new groupless systems show a rather marked difference from the indices of the previously known systems. It has been found convenient to list the indices of systems, so that those indices which contain the greatest number of sequences of period 1 appear at the left-hand side of the index table; that is, the indices are written in descending order of sequences of period 1. Systems with a group contain indices of the types 1^{12}, $1^8 2^2$,—but no groupless system has yet been found with an index containing more than six sequences of period 1—for example, $1^6 3^2$ contains the maximum number of sequences of period 1 found among the indices of the groupless systems. Hence the indices of the groupless type show a distinct shifting towards the right-hand side of the index table. This shift was to be expected since the new systems are headless and, therefore, the indices necessarily involve fewer sequences of period 1.

VASSAR COLLEGE.

NOTE ON GREEN'S THEOREM.

BY MR. C. A. EPPERSON.

(Read before the American Mathematical Society April 24, 1915.)

1. *Introduction.*—We wish in this paper to extend Green's theorem to apply to relations equivalent to the general linear partial differential equation of the second order in two variables. The relations are written in such a way as not to involve derivatives of the second order, and the theorem is proved without assuming their existence.* This point of view is desirable for the possibility of its application to physics.

2. *Statement of Lemma.*—We shall denote by a_{ij}, a_i, a, arbitrary continuous functions of x and y, the a_{ij} to have also continuous first and second partial derivatives, and the a_i to have continuous first partial derivatives, over a given simply connected region. By b_{ij}, b_i, b, we indicate the coefficients of the linear partial differential expression of the second order which is adjoint to the expression of which the coefficients are a_{ij}, a_i, a. That is, $b_{ij} = a_{ij}$,

$$b_i = 2\frac{\partial a_{i1}}{\partial x} + 2\frac{\partial a_{i2}}{\partial y} - a_i,$$

$$b = \frac{\partial^2 a_{11}}{\partial x^2} + \frac{\partial^2 a_{22}}{\partial y^2} + 2\frac{\partial^2 a_{12}}{\partial x \partial y} - \frac{\partial a_1}{\partial x} - \frac{\partial a_2}{\partial y} + a.$$

* The immediate occasion for the publication of this paper was the appearance of an article by C. W. Oseen, "Über einen Satz von Green und über die Definitionen von Rot und Div," *Rendiconti del Circolo Matematico di Palermo*, vol. 38 (1914), pp. 167–179, which applies this method, by means of the principles of vector analysis, to the special case of Poisson's equation. The general proof is a first step in the detailed working out of a general problem for partial differential equations (see preliminary communication of December, 1913, G. C. Evans, "Green's functions for linear partial differential expressions of the second order, and Green's theorem," BULLETIN, vol. 20, no. 6, March, 1914).

A proof has been given for equations of parabolic type, which may be generalized to the equations here considered. (G. C. Evans, "On the reduction of integro-differential equations," *Transactions Amer. Math. Society*, vol. 15, no. 4, p. 486, Oct., 1914. This paper also gives the literature of the problem). The proof referred to however, by a reduction, makes use of Green's theorem in its usual form; in the present paper, the result is reduced ab initio.

The b_{ij} are then continuous with their first and second partial derivatives, the b_i continuous with their first partial derivatives, and b continuous over the given region. We shall consider a square S_r, and by $\lim_{S_r \doteq p}$ we shall mean that no matter how small a circle we take about p the square shall ultimately become and remain entirely within that circle. We have then

LEMMA I.* *Let $u(x, y)$ and $v(x, y)$ and their first partial derivatives be continuous over the given region in the (xy) plane, and let p be a point within that region. If*

$$(1)\quad \begin{aligned}(f - bu)_p = \lim_{S_r \doteq p} \frac{1}{\sigma_r}\int_{S_r} &\left[\frac{\partial}{\partial x}(b_{11}u) + \frac{\partial}{\partial y}(b_{12}u) - b_1 u\right]dy \\ &- \left[\frac{\partial}{\partial x}(b_{12}u) + \frac{\partial}{\partial y}(b_{22}u) - b_2 u\right]dx\end{aligned}$$

and

$$(2)\quad \begin{aligned}(g - av)_p = \lim_{S_r \doteq p} \frac{1}{\sigma_r}\int_{S_r} &\left[\frac{\partial}{\partial x}(a_{11}v) + \frac{\partial}{\partial y}(a_{12}v) - a_1 v\right]dy \\ &- \left[\frac{\partial}{\partial x}(a_{12}v) + \frac{\partial}{\partial y}(a_{22}v) - a_2 v\right]dx;\end{aligned}$$

then

$$(3)\quad \begin{aligned}(vf - ug)_p = \lim_{S_r \doteq p} \frac{1}{\sigma_r}\int_{S_r} &\left[a_{11}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right)\right. \\ &\left. + a_{12}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \frac{a_1 - b_1}{2}uv\right]dy - \left[a_{21}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right)\right. \\ &\left. + a_{22}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \frac{a_2 - b_2}{2}uv\right]dx.\end{aligned}$$

3. *Proof of Lemma I.*—If we multiply equation (1) by v_p and equation (2) by u_p and subtract, we have, using the relations existing between the a's and b's of these equations,

---

* The reader will note that if the second derivatives of u and v exist the theorem corresponds to setting

$$f = a_{11}\frac{\partial^2 u}{\partial x^2} + a_{22}\frac{\partial^2 u}{\partial y^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_1\frac{\partial u}{\partial x} + a_2\frac{\partial u}{\partial y} + au$$

and g = the adjoint of f. We then have the ordinary Green's theorem.

$$
\begin{aligned}
&(vf - ug)_p + \frac{1}{2}\left[\frac{\partial}{\partial x}(a_1 - b_1) + \frac{\partial}{\partial y}(a_2 - b_2)\right]_p u_p v_p \\
&= \lim_{S_r \doteq p} \frac{1}{\sigma_r}\int_{S_r}\left[a_{11}\left(v_p \frac{\partial u}{\partial x} - u_p \frac{\partial v}{\partial x}\right) + a_{12}\left(v_p \frac{\partial u}{\partial y} - u_p \frac{\partial v}{\partial y}\right)\right. \\
(4)\qquad &\left. + \frac{a_1 - b_1}{2}(v_p u - u_p v)\right] dy \\
&- \left[a_{21}\left(v_p \frac{\partial u}{\partial x} - u_p \frac{\partial v}{\partial x}\right) + a_{22}\left(v_p \frac{\partial u}{\partial y} - u_p \frac{\partial v}{\partial y}\right)\right. \\
&\left. + \frac{a_2 - b_2}{2}\left(v_p u - u_p v\right)\right] dx.
\end{aligned}
$$

If now we consider the difference function D of the right-hand side of the equations (3) and (4), we have

$$
\begin{aligned}
\lim_{S_r \doteq p} D = \lim_{S_r \doteq p} \frac{1}{\sigma_r}\int_{S_r} &\left\{ a_{11}\left[(v_p - v_s)\frac{\partial u}{\partial x} - (u_p - u_s)\frac{\partial v}{\partial x}\right]\right. \\
&+ a_{12}\left[(v_p - v_s)\frac{\partial u}{\partial y} - (u_p - u_s)\frac{\partial v}{\partial y}\right] \\
&\left. + \frac{a_1 - b_1}{2}[(v_p - v_s)u_s + u_p v_s]\right\} dy \\
(5)\qquad &- \left\{ a_{21}\left[(v_p - v_s)\frac{\partial u}{\partial x} - (u_p - u_s)\frac{\partial v}{\partial x}\right]\right. \\
&+ a_{22}\left[(v_p - v_s)\frac{\partial u}{\partial y} - (u_p - u_s)\frac{\partial v}{\partial y}\right] \\
&\left. + \frac{a_2 - b_2}{2}[(v_p - v_s)u_s + u_p v_s]\right\} dx.
\end{aligned}
$$

In order to establish the lemma it is sufficient to show that

$$
\lim_{S_r \doteq p} D = \frac{1}{2}\left[\frac{\partial}{\partial x}(a_1 - b_1) + \frac{\partial}{\partial y}(a_2 - b_2)\right]_p u_p v_p.
$$

To obtain the value of D we will divide the square into four smaller rectangles by lines through p parallel to the sides of the square and consider the sum of the integrals extended around these smaller rectangles. For convenience of notation

we shall consider p as the origin, and the vertices of the square lettered as in accompanying figure (Fig. 1). We will write

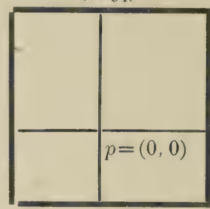


Fig. 1.

the curvilinear integral D_1 around one of these small rectangles, say $[(0, 0)/(x_1, y_1)]$, as a simple integral.

Thus

$$
\begin{aligned}
D_1 = \frac{1}{\sigma_r}\int_0^{y_1} \Bigg[\Bigg\{ & a_{11}\left[(v_p - v_s)\frac{\partial u}{\partial x} - (u_p - u_s)\frac{\partial v}{\partial x}\right] \\
& + a_{12}\left[(v_p - v_s)\frac{\partial u}{\partial y} - (u_p - u_s)\frac{\partial v}{\partial y}\right] \\
& + \frac{a_1 - b_1}{2}[(v_p - v_s)u_s + u_p v_s]\Bigg\}_{x_1, y} \\
& - \Bigg\{ a_{11}\left[(v_p - v_s)\frac{\partial u}{\partial x} - (u_p - u_s)\frac{\partial v}{\partial x}\right] \\
& + a_{12}\left[(v_p - v_s)\frac{\partial u}{\partial y} - (u_p - u_s)\frac{\partial v}{\partial y}\right] \\
& + \frac{a_1 - b_1}{2}[(v_p - v_s)u_s + u_p v_s]\Bigg\}_{0, y}\Bigg] dy
\end{aligned} \tag{6}
$$

plus the corresponding term to be integrated with respect to x.

If we note the figure we see that

$$\left|(v_s - v_p) - \left(\frac{\partial v_s}{\partial x}x_1 + \frac{\partial v_s}{\partial y}y\right)\right| < \epsilon(|x_1| + |y|) < 2\epsilon\delta$$

$$0 \leq y \leq y_1, \quad 0 \leq x \leq x_1, \quad \delta = \text{one side of the square},$$

if v_s has the coordinates (x_1, y), and a similar expression holds if v_s is on one of the other sides. Similar expressions hold

for $(u_s - u_p)$. Further, one infinitesimal ϵ may be used for all such expressions, as u and v and their first partial derivatives are uniformly continuous throughout the square. In the limit then $(v_s - v_p)$ may be replaced by $(\partial v_s/\partial x)x_1 + (\partial v_s/\partial y)y$ or by the corresponding expressions depending on the position of v_s.

If we carry out these substitutions in detail we obtain an expression of the form

$$
\begin{aligned}
D_1 = \frac{1}{\sigma_r}\int_0^{y_1}\Bigg[&\Bigg\{\left[a_{11}\left(\frac{\partial v}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\right]_{x_1, y} \\
&- \left[a_{11}\left(\frac{\partial v}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\right]_{0, y}\Bigg\}(-y) \\
&+ \Bigg\{\left[a_{12}\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)\right]_{x_1, y} \\
&- \left[a_{12}\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)\right]_{0, y}\Bigg\}(-x_1) \\
&+ \Bigg\{\left[\frac{a_1 - b_1}{2}\left(\frac{\partial v}{\partial y}u\right)\right]_{x_1, y} \\
&- \left[\frac{a_1 - b_1}{2}\left(\frac{\partial v}{\partial y}u\right)\right]_{0, y}\Bigg\}(-y) \\
&+ u_p\left[\left(\frac{a_1 - b_1}{2}v\right)_{x_1, y} - \left(\frac{a_1 - b_1}{2}v\right)_{0, y}\right] \\
&+ \left(\frac{a_1 - b_1}{2}\frac{\partial v}{\partial x}u\right)_{x_1, y}(-x_1)\Bigg]dy
\end{aligned} \tag{7}
$$

plus exactly similar terms to be integrated with respect to x, plus terms every one of which is in absolute value less than ϵ times a constant less than $|\sigma_r|$.

If we apply the law of the mean every term, except the last two, is of one of the forms

$$(a)\qquad [P(x_1, y) - P(0, y)]\left(\frac{x_1y_1}{\sigma_r}\right) \text{ or } [P(x_1, y) - P(0, y)]\left(\frac{x_1x_1}{\sigma_r}\right)$$

$$(b)\qquad \epsilon\left(\frac{x_1y_1}{\sigma_r}\right) \quad\text{or}\quad \epsilon\left(\frac{x_1x_1}{\sigma_r}\right), \text{ etc.,}$$

where $P(x, y)$ is uniformly continuous in x and in y throughout the square. Hence in the limits every term except the last two vanishes and we have

$$D_1 = \frac{1}{\sigma_r}\left[\left(\frac{a_1 - b_1}{2}\frac{\partial v}{\partial x}u\right)_{x_1,\, y}(-x_1) + u_p\frac{\partial}{\partial x}\left(\frac{a_1 - b_1}{2}v\right)_{0,\, y}(x_1)\right]y_1 + \eta, \tag{8}$$

where $\lim_{S_r \doteq p} \eta = 0$,

$$= \frac{1}{\sigma_r}u_p\left[\left(\frac{a_1 - b_1}{2}\frac{\partial v}{\partial x}\right)_{x_1,\, y}(-x_1) + \left(\frac{a_1 - b_1}{2}\frac{\partial v}{\partial x}\right)_{0,\, y}(x_1) + \left(v\frac{\partial}{\partial x}\frac{a_1 - b_1}{2}\right)_{0,\, y}(x_1)\right]y_1 + \eta + \eta',$$

where

$$|\eta'| < \epsilon\left|\frac{a_1 - b_1}{2}\frac{\partial v}{\partial x}\right|,$$

which gives us, if we combine with it the corresponding terms from the integration with respect to x,

$$D_1 = \left(\frac{x_1 y_1}{\sigma_r}\right)\frac{1}{2}\left[\frac{\partial}{\partial x}(a_1 - b_1) + \frac{\partial}{\partial y}(a_2 - b_2)\right]_p u_p v_p + \omega,$$

where $\lim_{S_r \doteq p} \omega = 0$, ω being less in absolute value than $\epsilon\delta^2/\sigma_r$ times some constant.

If we now carry the integration around the other rectangles we have similar expressions for D_2, D_3, and D_4.

Since $(x_1y_1 + x_2y_1 + x_2y_2 + x_1y_2)/\sigma_r = 1$, we have

$$\lim_{S_r \doteq p} D = \frac{1}{2}\left[\frac{\partial}{\partial x}(a_1 - b_1) + \frac{\partial}{\partial y}(a_2 - b_2)\right]_p u_p v_p.$$

The lemma is therefore established.

4. *Statement of Green's Theorem.*—By a *standard curve* we shall mean a closed regular curve which does not cut itself and which cannot be cut in more than a finite number of points by any horizontal or vertical straight line.

We may now state Green's theorem.

THEOREM I. *If $u(x, y)$ and $v(x, y)$ with their first partial derivatives, and f and g, are continuous throughout a simply*

connected region, and if for every standard curve S, contained wholly within that region,*

$$\begin{aligned}(9)\quad \iint_\sigma (f - bu)dxdy = \int_S \left[\frac{\partial}{\partial x}(b_{11}u) + \frac{\partial}{\partial y}(b_{12}u) - b_1 u\right] dy \\ - \left[\frac{\partial}{\partial x}(b_{21}u) + \frac{\partial}{\partial y}(b_{22}u) - b_2 u\right] dx\end{aligned}$$

and

$$\begin{aligned}(10)\quad \iint_\sigma (g - av)dxdy = \int_S \left[\frac{\partial}{\partial x}(a_{11}v) + \frac{\partial}{\partial y}(a_{12}v) - a_1 v)\right] dy \\ - \left[\frac{\partial}{\partial x}(a_{21}v) + \frac{\partial}{\partial y}(a_{22}v) - a_2 v\right] dx;\end{aligned}$$

then

$$\begin{aligned}(11)\quad \iint_\sigma (vf - ug)dxdy = \int_S \Bigg[a_{11}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) \\ + a_{12}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \frac{a_1 - b_1}{2} uv\Bigg] dy \\ - \Bigg[a_{21}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + a_{22}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) \\ + \frac{a_2 - b_2}{2} uv\Bigg] dx.\end{aligned}$$

Proof of Theorem I.—For convenience in writing our equations we will set

$$A_i = a_{i1}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + a_{i2}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \frac{a_i - b_i}{2} uv.$$

Now divide the region enclosed by the contour into a number of squares σ_i of side δ. Then

$$\iint_\sigma (vf - ug)dxdy = \sum^n \iint_{\sigma_i} (vf - ug)dx_i dy_i + \eta \quad (n > N),$$

where η is the integral over that portion of σ not contained in $\sum^n \sigma_i$ and can be made as small as we please by choosing n sufficiently large. By the law of the mean

* The conclusion holds for the boundary of the region if it is a standard curve.

$$\int\int_{\sigma_i}(vf - ug)dx_i dy_i = (vf - ug)_{p_i}\sigma_i,$$

where p_i is some point in σ_i. Then

$$\int\int_{\sigma}(vf - ug)dxdy = \sum^{n}(vf - ug)_{p_i}\sigma_i + \eta \quad (n > N).$$

The hypotheses of this theorem imply the hypotheses of Lemma I. Hence, by that lemma,

$$(vf - ug)_{p_i} = \lim_{S_i \doteq p_i}\frac{1}{\sigma_i}\int_{S_i} A_1 dy - A_2 dx,$$

so that

$$(12) \quad \begin{aligned} \int\int_{\sigma}(vf - ug)dxdy &= \sum^{n}(vf - ug)_{p_i}\sigma_i + \eta \\ &= \sum^{n}\sigma_i \lim_{S_i \doteq p_i}\frac{1}{\sigma_i}\int_{S_i} A_1 dy - A_2 dx. \end{aligned}$$

We have to do with a uniform limit in the right-hand member of the above equation, since on account of the assumed continuity of u, $\partial u/\partial x$, $\partial u/\partial y$ the quantity ϵ which occurs in the proof of Lemma I depends merely on the size of the square, and not on the position of the point p. Hence

$$\lim_{S_i \doteq p_i}\frac{1}{\sigma_i}\int_{S_i} A_1 dy - A_2 dx$$

differs from

$$\frac{1}{\sigma_i}\int_{S_i} A_1 dy - A_2 dx$$

by less than $\bar{\epsilon}$, where $\lim_{\delta \doteq 0}\bar{\epsilon} = 0$, and

$$\sum^{n}\sigma_i \lim_{S_i \doteq p_i}\frac{1}{\sigma_i}\int_{S_i} A_i dy - A_2 dx$$

differs from

$$\int_{S_r} A_1 dy - A_2 dx$$

by less than $\bar{\epsilon}\sigma$, where S_r is the contour formed by the exterior boundary of the approximating squares. It then follows that

$$\int\int_{\sigma}(vf - ug)dxdy = \int_{S_r} A_1 dy - A_2 dx + \eta',$$

where $\lim_{\delta \doteq 0} \eta' = 0$. But since A_1 and A_2 are uniformly continuous in x and y throughout the region, we have

$$\lim_{\delta \doteq 0} \int_{S_r} A_1 dy - A_2 dx + \eta' = \int_S A_1 dy - A_2 dx,$$

and therefore finally

$$\int\int_\sigma (vf - ug) dx dy = \int_S A_1 dy - A_2 dx,$$

which was to be proved.

Corollary I. *Theorem for Poisson's Equation. If u and v and their first partial derivatives, and f and g, are continuous throughout a simply connected region, and if for every standard curve S contained wholly within that region,*

$$\int\int_\sigma f(x, y) dx dy = \int_S \frac{\partial u}{\partial n} ds$$

and

$$\int\int_\sigma g(x, y) dx dy = \int_S \frac{\partial v}{\partial n} ds$$

then

$$\int\int_\sigma (vf - ug) dx dy = \int_S \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds.$$

If we put in this $v = 1/r$ we get the well-known formula due to Green.

Corollary II. By $\lim_{S \doteq p}$ we shall mean that the standard curve S approaches p uniformly, i. e., if we take a circle with center at p and of arbitrarily small radius the standard curve S ultimately becomes and remains within the circle. We may now state

Corollary II. *If u and v and their first partial derivatives are continuous within and on the boundary of a region enclosed by a standard curve S in the (xy) plane, and if*

$$(f - bu)_p = \lim_{S \doteq p} \frac{1}{\sigma} \int_S \left[\frac{\partial}{\partial x} (b_{11} u) + \frac{\partial}{\partial y} (b_{22} u) - b_2 u \right] dy - \left[\frac{\partial}{\partial x} (b_{12} u) + \frac{\partial}{\partial y} (b_{22} u) - b_2 u \right] dx$$

and

$$(g - av)_p = \lim_{S \doteq p} \frac{1}{\sigma} \int_S \left[\frac{\partial}{\partial x}(a_{11}v) + \frac{\partial}{\partial y}(a_{12}v) - a_1 v \right] dy - \left[\frac{\partial}{\partial x}(a_{21}v) + \frac{\partial}{\partial y}(a_{22}v) - a_2 v \right] dx,$$

then

$$(vf - ug)_p = \lim_{S \doteq p} \frac{1}{\sigma} \int_S \left[a_{11}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} \right) + a_{12}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y} \right) + \frac{a_1 - b_1}{2} uv \right] dy - \left[a_{21}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} \right) + a_{22}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y} \right) + \frac{a_2 - b_2}{2} uv \right] dx,$$

where p is any point within S.

THE RICE INSTITUTE,
February, 1915.

CONVERGENCE OF THE SERIES $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{x^i y^j}{i - j\gamma}$ (γ IRRATIONAL).

BY PROFESSOR W. D. MACMILLAN.

(Read before the American Mathematical Society April 2, 1915.)

The method of proof which is here used depends upon the properties of continued fractions. Any irrational number γ can be expanded as a simple continued fraction

$$\gamma = a_1 + \frac{1}{a_2 +}\frac{1}{a_3 +}\frac{1}{a_4 +}\cdots.$$

Let p_n/q_n be the nth principal convergent,* and P/Q be any intermediate convergent lying between p_{n-2}/q_{n-2} and p_n/q_n. Then

$$\frac{p_{n-2}}{q_{n-2}} < \frac{P}{Q} < \frac{p_n}{q_n} < \gamma < \frac{p_{n+1}}{q_{n+1}} < \frac{p_{n-1}}{q_{n-1}}$$

if n is odd, and

* The notation used here agrees with that of Chrystal's Algebra, Vol. II, Chap. XXXII.

$$\frac{p_{n-2}}{q_{n-2}} > \frac{P}{Q} > \frac{p_n}{q_n} > \gamma > \frac{p_{n+1}}{q_{n+1}} > \frac{p_{n-1}}{q_{n-1}}$$

if n is even.

For n either even or odd

$$\left|\frac{P}{Q} - \gamma\right| > \left|\frac{P}{Q} - \frac{p_n}{q_n}\right| = \left|\frac{p_{n-2} + kp_{n-1}}{q_{n-2} + kq_{n-1}} - \frac{p_n}{q_n}\right|$$
$$1 \leqq k \leqq (a_n - 1),$$

from which is derived

$$\left|\frac{P}{Q} - \gamma\right| > \frac{a_n - k}{q_n Q}.$$

Since $q_n = q_{n-2} + a_n q_{n-1}$ and $Q = q_{n-2} + kq_{n-1}$, we have

$$q_n = Q + (a_n - k)q_{n-1} < Q + a_n q_{n-1} < (a_n + 1)Q.$$

Hence

$$\left|\frac{P}{Q} - \gamma\right| > \frac{a_n - k}{q_n Q} \geqq \frac{1}{q_n Q} > \frac{1}{Q^2(a_n + 2)},$$

and likewise

$$\left|\frac{p_n}{q_n} - \gamma\right| > \frac{1}{q_n(q_{n+1} + q_n)} > \frac{1}{q_n{}^2(a_{n+1} + 2)}.$$

We have then

$$(1) \qquad | P - Q\gamma | > \frac{1}{Q(a_n + 2)}$$

and

$$| p_{n-1} - q_{n-1}\gamma | > \frac{1}{q_{n-1}(a_n + 2)}.$$

Let us suppose now that $a_n + 2 < M$ for every n, an hypothesis which is certainly satisfied by every simple quadratic surd $(m \pm \sqrt{n})/l$, where l, m and n are integers. Then if P/Q is any convergent, principal or intermediate, it follows from (1) that

$$(2) \qquad | P - Q\gamma | > \frac{1}{MQ}.$$

We shall show now that for any two integers whatever, i and j,

(3) $$| i - j\gamma | > \frac{1}{Mj}.$$

Let P_1/Q_1 and P_2/Q_2 be two successive convergents (principal or intermediate) such that $Q_1 \leqq j \leqq Q_2$. From the general theory of continued fractions it is known that P_1/Q_1 and P_2/Q_2 are closer approximations to γ than any other rational fractions whose denominators are less than Q_2. Consequently

$$\left| \frac{i}{j} - \gamma \right| > \left| \frac{P_1}{Q_1} - \gamma \right| > \frac{1}{MQ_1^2},$$

and therefore, since $j > Q_1$,

(4) $$| i - j\gamma | > | P_1 - Q_1\gamma | > \frac{1}{MQ_1} > \frac{1}{Mj}.$$

Consider now the series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^i y^j}{i - j\gamma} \qquad (i + j > 0),$$

where γ is an irrational number which, when expressed as a simple continued fraction, satisfies the condition that $a_n + 2 < M$ for every n. Then we will have $| i - j\gamma | > 1/(Mj)$ and consequently

$$\frac{1}{| i - j\gamma |} < Mj;$$

so that

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^i y^j}{i - j\gamma} = \sum_{i=1}^{\infty} \frac{x^i}{i} + \sum_{i=0}^{\infty} x^i \sum_{j=1}^{\infty} \frac{y^j}{i - j\gamma} \ll \sum_{i=1}^{\infty} x^i$$
$$+ M \sum_{i=1}^{\infty} x^i \sum_{j=1}^{\infty} j y^j \ll \frac{x}{1 - x} + \frac{M}{1 - x} \cdot \frac{y}{(1 - y)^2},$$

which converges if $| x | < 1$ and $| y | < 1$. Therefore the series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^i y^j}{i - j\gamma} \qquad (i + j > 0)$$

converges if $| x |$ and $| y |$ are both less than unity, which is somewhat remarkable in that the denominators, which have no lower limit, impose no restriction upon the radii of convergence of the series.

The restrictions upon γ in the above discussion can be considerably reduced. It is seen from (1), if P/Q is any convergent (principal or intermediate), that

$$| P - Q\gamma | > \frac{1}{Q(a_n + 2)},$$

where a_n is the first partial quotient above Q. Let us suppose now that

$$(5) \quad a_n + 2 < M(q_{n-1} + 1)(q_{n-1} + 2) \cdots (q_{n-1} + S - 1),$$

where S is any positive integer independent of n. Then, since $Q > q_{n-1}$,

$$a_n + 2 < M(Q + 1) \cdots (Q + S - 1),$$

so that

$$| P - Q\gamma | > \frac{1}{MQ(Q + 1) \cdots (Q + S - 1)}.$$

Then, just as before, if i and j are any two integers such that $Q \leqq j \leqq Q_1$, we shall have

$$| i - j\gamma | > | P - Q\gamma | > \frac{1}{MQ(Q + 1) \cdots (Q + S - 1)}$$

$$> \frac{1}{Mj(j + 1) \cdots (j + S - 1)}$$

and also

$$\frac{1}{| i - j\gamma |} < Mj(j + 1) \cdots (j + S - 1).$$

If then γ satisfies these new conditions the series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^i y^j}{i - j\gamma} \ll \frac{x}{1 - x} + M \sum_{i=0}^{\infty} x^i y \sum_{j=1}^{\infty} j(j+1) \cdots (j+S-1) y^{j-1}.$$

But since

$$\sum_{j=1}^{\infty} j(j + 1) \cdots (j + S - 1) y^{j-1} = \frac{d^S}{dy^S}\left(\frac{1}{1 - y}\right) = \frac{S!}{(1 - y)^{S+1}}$$

we have

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^i y^j}{i - j\gamma} \ll \frac{x}{1 - x} + \frac{yMS!}{(1 - x)(1 - y)^{S+1}}$$

and therefore convergent provided $|x|$ and $|y|$ are both less than unity.

Corollary.—If the series $f(x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}x^iy^j \quad (i + j > 0)$ converges for $|x| < 1/\xi$, $|y| < 1/\eta$, so that

$$f(x, y) \ll \frac{N}{(1 - \xi x)(1 - \eta y)} - N,$$

and if γ is an irrational number which satisfies (5), then the series

$$F(x, y) = \sum_{1=0}^{\infty}\sum_{j=0}^{\infty} \frac{a_{ij}}{i - j\gamma} x^iy^j \qquad (i + j > 0),$$

converges provided $|x| < 1/\xi$ and $|y| < 1/\eta$. Furthermore

$$F(x, y) \ll \frac{N}{1 - \xi x}\left[\xi x + \frac{MS!}{(1 - \xi x)(1 - \eta y)^{S+1}}\right].$$

It will perhaps be interesting to note the character of the condition that $a_{n+1} + 2 < Mq_n(q_n + 1)\cdots(q_n + S - 1)$. Let us suppose that $a_n = n!$. It is found then $q_{n-1} = (n - 1)!(n - 2)!\cdots 2! + \cdots$. It is sufficient then to take $M = 1$, $S = 2$, in order to satisfy the condition. If we suppose that $a_n = 10^{10^{n-1}}$ we find that $q_{n-1} = 10^{10^{n-2}}\cdot 10^{10^{n-3}}\cdot\ \cdots\ 10^{10^0} + \cdots$, and it is sufficient to take $M = 10$, $S = 10$. If however we suppose that $a_n = 10^{n!}$ then $q_{n-1} = 10^{(n-1)!+(n-2)!+\cdots+2!} + \cdots$, and there do not exist an M and an S which satisfy the condition.

Application of these Series.—(*a*) The function

$$W = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\lambda^m\mu^n}{m - nz}, \quad |\lambda| < 1, \quad |\mu| < 1,$$

where z is a complex variable, is a holomorphic function of z everywhere except in the neighborhood of the positive real axis, which is a line of essential singularities. Nevertheless the value of the function is finite for those real, positive irrational values of z which satisfy the above condition; furthermore the function is continuous across the real axis at any one of these points. To show the continuity, let $z = \gamma$ be such a point and let $z = \gamma + t\cos\alpha + it\sin\alpha$ be a

straight line which crosses the real axis at γ making an angle α with the real axis. Then

$$W = \Sigma\Sigma \frac{\lambda^m\mu^n}{(m - n\gamma - nt\cos\alpha) - i(nt\sin\alpha)}$$

$$= \Sigma\Sigma \frac{[m - n\gamma - nt\cos\alpha] + i(nt\sin\alpha)}{[m - n\gamma - nt\cos\alpha]^2 + n^2t^2\sin^2\alpha}\lambda^m\mu^n.$$

If now we write $W = W_1 + iW_2$, and for brevity suppose λ and μ real, we have

$$W_1 = \Sigma\Sigma\lambda^m\mu^n \frac{m - n\gamma - nt\cos\alpha}{[m - n\gamma - nt\cos\alpha]^2 + n^2t^2\sin^2\alpha},$$

$$W_2 = \Sigma\Sigma\lambda^m\mu^n \frac{nt\sin\alpha}{[m - n\gamma - nt\cos\alpha]^2 + n^2t^2\sin^2\alpha}.$$

Consider now

$$\frac{nt\sin\alpha}{[m - n\gamma - nt\cos\alpha]^2 + n^2t^2\sin^2\alpha}.$$

As a function of the variable t this expression has a maximum or a minimum for $n^2t^2 = (m - n\gamma)^2$. It has a maximum equal to

$$\frac{1}{(m - n\gamma)[(1 + \cos\alpha)^2 + \sin^2\alpha]}$$

for $nt = (m - n\gamma)$, and a minimum equal to

$$\frac{-1}{(m - n\gamma)[(1 + \cos\alpha)^2 + \sin^2\alpha]}$$

for $nt = -(m - n\gamma)$. Consequently

$$W_2 \ll \frac{|\sin\alpha|}{[(1 - \cos\alpha)^2 + \sin^2\alpha]}\cdot\Sigma\Sigma\frac{\lambda^m\mu^n}{|m - n\gamma|},$$

which is absolutely convergent. Whence W_2, and in the same manner W_1, is absolutely and uniformly convergent for all real values of t. Consequently W is a continuous function of z all along this straight line.

(*b*) Consider the linear partial differential equation

$$x_1\frac{\partial\phi}{\partial x_1} - \gamma x_2\frac{\partial\phi}{\partial x_2} = p_1\phi + p_2,$$

where γ is a positive irrational number which satisfies the condition $a_{n+1} + 2 < Mq_n(q_n + 1) \cdots (q_n + S - 1)$, and $p_1 = \Sigma\Sigma a_{ij} x_1^i x_2^j$, $p_2 = \Sigma\Sigma b_{ij} x_1^i x_2^j$, are two convergent power series in x_1 and x_2.

We will take first the homogeneous equation

$$x_1 \frac{\partial \phi}{\partial x_1} - \gamma x_2 \frac{\partial \phi}{\partial x_2} = p_1 \phi,$$

and put $\psi = \log \phi$. Then

$$x_1 \frac{\partial \psi}{\partial x_1} - \gamma x_2 \frac{\partial \psi}{\partial x_2} = p_1.$$

The solution of this equation is

$$\psi = \Sigma\Sigma \frac{a_{ij}}{i - j\gamma} x_1^i x_2^j + \text{an arbitrary function of } (x_1^\gamma x_2),$$

and by the above corollary this series has the same region of validity as p_1 itself. It follows therefore that $\phi = e^\psi$ also is a convergent power series in x_1 and x_2, if the arbitrary function is taken equal to zero.

Returning now to the equation

$$x_1 \frac{\partial \phi}{\partial x_1} - \gamma x_2 \frac{\partial \phi}{\partial x_2} = p_1 \phi + p_2,$$

let us take $\phi = \omega e^\psi$, where e^ψ is the function already determined, and ω is an unknown function. We have then

$$x_1 \frac{\partial \omega}{\partial x_1} - \gamma x_2 \frac{\partial \omega}{\partial x_2} = p_2 e^{-\psi} = \Sigma\Sigma c_{ij} x_1^i x_2^j,$$

where $\Sigma\Sigma c_{ij} x_1^i x_2^j$ is the expansion of $p_2 e^{-\psi}$ and is therefore a convergent series. The solution of this equation is

$$\omega = \Sigma\Sigma \frac{c_{ij}}{i - j\gamma} x_1^i x_2^j + \text{an arbitrary function of } (x_1^\gamma x_2),$$

which likewise is a convergent series. Consequently $\phi = (A + \omega)e^\psi$, where A is an arbitrary function of $(x_1^\gamma x_2)$, is a solution of the differential equation.

THE UNIVERSITY OF CHICAGO,
May 15, 1915.

A CERTAIN CLASS OF FUNCTIONS CONNECTED WITH FUCHSIAN GROUPS.

BY PROFESSOR ARNOLD EMCH.

(Read before the American Mathematical Society April 24, 1915.)

1. CONSIDER a Fuchsian group Γ of linear substitutions

$$V_i \equiv z_i = \frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i} \qquad (i = 1, 2, 3, \cdots) \tag{1}$$

$$\alpha_i \delta_i - \beta_i \gamma_i = 1,$$

that transform the unit circle into itself, and for which the unit circle is a natural boundary. The index i for which z_i approaches a point of the boundary we denote by ∞, so that $\lim_{i=\infty} (z_i) = e^{i\phi}$, where ϕ may have any value from 0 to 2π. Let $z_0 = z$ represent identity. Denote by $R_0 = R$ the fundamental region in which z lies, and by $R_1, R_2, \cdots$ the regions resulting from R by the substitutions V_i $(i = 1, 2, 3, \cdots)$. Let e_i be the greatest "elongation" of the boundary of R_i, i. e., the maximum distance between two points of the boundary of R_i; then, according to a theorem due to Bricard,* it is possible to circumscribe a circle C_i to the region R_i, such that its radius does not need to be greater than at most $e_i/\sqrt{3}$.

For $i \neq \infty$, the area A_i of R_i, being that of a singly connected region bounded by circular arcs, is finite, so that for the ratio of the area of the circle C_i to that of the region R_i we have

$$1 < \frac{\pi e_i^2}{3A_i} < M \qquad (i = 1, 2, 3, \cdots), \tag{2}$$

where M is a positive finite quantity > 1. But it can be shown that this inequality also exists when $\lim (i) = \infty$, or $\lim_{=\infty} (z_i) = e^{i\phi}$. Hence from (2) we get

$$3\Sigma A_i < \Sigma \pi e_i^2 < 3M\Sigma A_i, \tag{3}$$

* "Théorèmes sur les courbes et les surfaces fermées," *Nouvelles Annales Je Mathématique*, 4. ser., vol. 14, pp. 19–25 (January, 1914).

in which the sums are extended over the whole group Γ. As $\Sigma A_i = \pi$ is a finite quantity we find that the sum of the areas of all circles C_i, and consequently the sum of the squares of the radii of all these circles is finite.

2. Choose now within R any two points a and b and a variable point z, so that the area formed by the euclidean triangle $z_i a_i b_i$ lies entirely within C_i. Now

$$| z_i - a_i | \leqq e_i; \ | z_i - b_i | \leqq e_i,$$

hence

$$| z_i - a_i | \cdot | z_i - b_i | \leqq e_i^2,$$

and

$$\sum_{i=0}^{\infty} | z_i - a_i | \cdot | z_i - b_i | \leqq \sum_{i=0}^{\infty} e_i^2. \tag{4}$$

But, according to (3), $\sum_{i=0}^{\infty} e_i^2$ is a finite quantity. The left side of (4) is therefore an absolutely convergent series, for all values of z within R. The condition for uniform convergence within the whole domain is evidently also satisfied, so that we can state

THEOREM I. *The series*

$$\sum_{i=0}^{\infty} (z_i - a_i)(z_i - b_i)$$

extended over a Fuchsian group Γ, *with the unit circle as a natural boundary and* z, a, b *lying within the fundamental region of* Γ, *is a uniformly convergent series, and defines an analytic function within* R *that vanishes for* $z = a$ *and* $z = b$ *and has no infinities within* R. *The result is still valid when* $z_b = z_a$, *so that*

$$\sum_{i=0}^{\infty} (z_i - a_i)^2$$

also defines such a function which at $z = a$ *has a zero of the second order.*

3. The theorem may immediately be generalized. Choose for z and a any two points within the unit circle (excluding the boundary). The straight line joining them is cut by a finite number of polygons R_i into the segments $l_1, l_2, l_3, \cdots, l_r$. Any substitution $V_\lambda \equiv \begin{pmatrix} \alpha_\lambda \beta_\lambda \\ \gamma_\lambda \delta_\lambda \end{pmatrix}$ of the group Γ transforms the

straight segment from z to a into an arc of a circle from z_λ to a_λ and the segments l_i into arcs $l_{i\lambda}$ intercepted by the corresponding polygons arising from the substitution V_λ. Every arc $l_{i\lambda}$ is subtended by a chord $s_{i\lambda} < l_{i\lambda} < e_{i\lambda}$, where $e_{i\lambda}$ denotes the elongation of the polygon (region) $R_{i\lambda}$. From this follows immediately that

$$f_\lambda = |z_\lambda - a_\lambda| < e_{1\lambda} + e_{2\lambda} + \cdots + e_{r\lambda},$$

and

$$(f_\lambda)^2 < (e_{1\lambda} + e_{2\lambda} + \cdots + e_{i\lambda} + \cdots + e_{k\lambda} + \cdots + e_{r\lambda})^2.$$

From the inequality

$$2e_{i\lambda}\, e_{k\lambda} < e_{i\lambda}{}^2 + e_{k\lambda}{}^2$$

we derive without difficulty

$$(5) \qquad 2 \sum_{\substack{i=1,\, k=1 \\ i \neq k}}^{r} e_{i\lambda} e_{k\lambda} < (r - 1)(e_{1\lambda}{}^2 + e_{2\lambda}{}^2 + \cdots + e_{r\lambda}{}^2).$$

Now

$$\sum_{\lambda=0}^{\infty} (f_\lambda)^2 = \sum_{\lambda=0}^{\infty} (e_{1\lambda}{}^2 + e_{2\lambda}{}^2 + \cdots + e_{r\lambda}{}^2) + 2 \sum_{\lambda=0}^{\infty} \sum_{\substack{i=1,\, k=1 \\ 1 \neq k}}^{r} e_{i\lambda} e_{k\lambda};$$

hence, according to (5),

$$(6) \qquad \sum_{\lambda=0}^{\infty} (f_\lambda)^2 < r \sum_{\lambda=0}^{\infty} (e_{1\lambda}{}^2 + e_{2\lambda}{}^2 + \cdots + e_{r\lambda}{}^2).$$

But

$$\sum_{\lambda=0}^{\infty} e_{i\lambda}{}^2 = \sum_{\lambda=0}^{\infty} e_{k\lambda}{}^2,$$

so that (6) reduces to

$$(7) \qquad \sum_{\lambda=0}^{\infty} (f_\lambda)^2 < r^2 \sum_{\lambda=0}^{\infty} e_\lambda{}^2.$$

The right side of this inequality is a finite quantity, so that the series on the left side is absolutely convergent. Hence

THEOREM II. *The series*

$$\sum_{\lambda=0}^{\infty} (z_\lambda - a_\lambda)^2$$

extended over a Fuchsian group with the unit circle as a natural boundary, where z and a are any two points within the unit circle and not on the boundary, when a is fixed, is an absolutely and uniformly convergent series of z for all points within and not on the boundary, and represents an analytic function in the neighborhood of all such points. It has a zero of the second order for $z = a$, and has the unit circle as a natural boundary.

4. This theorem admits of a further generalization. Choose any three points z, z', a within and not on the unit circle, and write $f_\lambda = |z_\lambda - a_\lambda|$, $g_\lambda = |z_\lambda' - a_\lambda|$. Assuming $f \neq 0$ $g \neq 0$, it is possible to find a positive finite number M such that the ratio $g_\lambda/f_\lambda < M$, $\lambda = 1, 2, 3, \cdots$, also when z_λ approaches a point on the unit circle. We have therefore $g_\lambda < Mf_\lambda$, and

$$f_\lambda g_\lambda < Mf_\lambda^2,$$

and consequently

$$\sum_{\lambda=0}^{\infty} f_\lambda g_\lambda < M \sum_{\lambda=0}^{\infty} f_\lambda^2. \tag{8}$$

As the right side of this inequality is absolutely convergent, it follows that

$$\sum_{\lambda=0}^{\infty} f_\lambda g_\lambda$$

is an absolutely convergent series, and that consequently

$$\sum_{\lambda=0}^{\infty} (z_\lambda - a_\lambda)(z_\lambda' - a_\lambda) \tag{9}$$

is absolutely and uniformly convergent, and, for a and z' constant, defines an analytic function of z for all points within and not on the boundary of the unit circle. It vanishes for $z = a$ and has the unit circle as a natural boundary. Nothing is lost in the convergency proof of (9) by assuming z and z' fixed and a as variable. Hence putting in (9) $z = a$, $z' = b$ and $a = z$ we may state

Theorem III. *The series*

$$\sum_{\lambda=0}^{\infty} (z_\lambda - a_\lambda)(z_\lambda - b_\lambda),$$

where a and b are any two points within and not on the unit circle, is absolutely and uniformly convergent and represents an analytic

function of z within the unit circle, which is a natural boundary of the function. It has $z = a$ and $z = b$ as zeros.

5. Making use of the proposition that for an analytic function $F(z)$ which within a certain region has the character of a rational function, such that for any point z_0 of this region $F(z_0)$ exists,

$$\lim_{z=z' \doteq z_0} \left(\frac{F(z) - F(z')}{z - z'} \right) = F'(z_0) \tag{10}$$

we may extend theorem III to an even more general type of functions. Let $\mathfrak{R}(z)$ be a rational function of z which for $z = 0$ does not become infinite. Putting $(z_\lambda - a_\lambda)(z_\lambda - b_\lambda) = u_\lambda$, $(z_\lambda' - a_\lambda')(z_\lambda' - b_\lambda') = u_\lambda'$, where z', a', b' denote a set like z, a, b, then as z_λ ($\lambda = \infty$) approaches a definite point $e^{i\phi}$ on the boundary a_λ, b_λ, a_λ', b_λ', z_λ' will approach the same point, and u and u' will approach zero as a limit. Consequently

$$\lim_{z_\lambda \doteq e^{i\phi}} \left\{ \frac{\mathfrak{R}(u_\lambda) - \mathfrak{R}(u_\lambda')}{u_\lambda - u_\lambda'} \right\} = \mathfrak{R}'(0) \tag{11}$$

is a finite quantity, and as $\Sigma(u - u')$ is absolutely and uniformly convergent, also $\sum_{\lambda=0}^{\infty} \{\mathfrak{R}(u) - \mathfrak{R}(u')\}$ will be absolutely and uniformly convergent within the unit circle, except for a finite number of values of u and u', which are poles of $\mathfrak{R}(u)$, and their congruents in the group Γ.* Hence, with the expressions for u, u' and $\mathfrak{R}$ defined as above, we may state

THEOREM IV. *When a, b, a', b', z' are fixed, so that no u_λ' is a pole of $\mathfrak{R}(z)$, then*

$$\sum_{\lambda=0}^{\infty} [\mathfrak{R}\{(z_\lambda - a_\lambda)(z_\lambda - b_\lambda)\} - \mathfrak{R}\{(z_\lambda' - a_\lambda')(z_\lambda' - b_\lambda')\}]$$

extended over the whole Fuchsian group represents an analytic function of z, which has the same poles as those of $\mathfrak{R}(u)$ and their congruents, and which has the unit circle as a natural boundary.

It appears that in general these functions are not automorphic in the ordinary sense.

UNIVERSITY OF ILLINOIS.

---

* For a statement of formula (10) and its applications to trigonometric and elliptic functions see Schottky: "Ueber die Funktionenklasse, die der Gleichung $F\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) = F(x)$ genügt"; *Crelle*, vol. 143 (1913), pp. 1–24.

PROFESSOR BÔCHER'S VIEWS CONCERNING THE GEOMETRY OF INVERSION.

BY PROFESSOR EDUARD STUDY.

IN a recent paper (this BULLETIN, volume 20, pages 185–200, January, 1914), Professor M. Bôcher sets forth what he thinks are sound principles for dealing with geometry, and more especially with the geometry of inversion. Herein I am glad to agree with him. The same and similar principles have been expounded, and applied, in a less elementary (but more comprehensive) manner by myself. Professor Bôcher's article may be looked upon as commenting on the import of the conception of *natural continua* introduced by me in 1903,* though he makes no use of this notion itself.

There seems to be, however, little agreement in other respects. Professor Bôcher quotes one of my articles ("Das Apollonische Problem," *Mathematische Annalen*, volume 49, 1897), merely stating that it is long and yet does not contain a word concerning the "region at infinity."† The first of these assertions is right as a matter of course, assuming a suitably chosen standard of length, and as to the second I will not quarrel with my critic. What I am concerned with is merely the inference which the author leaves to his readers. This inference would appear to be that I had been thoughtless, or careless, or regardless enough to publish my results in an embryonic state of development. It will inevitably be understood that all my theorems are "true in general" only, and consequently incorrect. I am found guilty of having committed an error to which, in recent years, I have myself objected often and strongly. I am caught in my own trap, and no mistake. There will be possibly some people who will enjoy this. But it cannot reasonably be expected that I should be one of their number. Therefore I beg to point out a few trifling circumstances that apparently have not been appreciated by my critic.

1. Let us supply the missing (but certainly indispensable)

---

* In my book, Geometrie der Dynamen, §§ 27, 28. This seems to have escaped Professor Bôcher's notice. He might have been aware of it, though, for he found a reference in connection with his own topic in a paper by H. Beck.

† Bôcher's own term is "the infinite region."

definitions, say, by the simple expedient of inserting a reference to Professor Bôcher's book of 1894. Then the haziness to which my critic objects will disappear at once and throughout. This remark, obvious as it is, does not seem to have occurred to Professor Bôcher; otherwise he ought to have mentioned it, for it alters the matter considerably. Probably he and I wrote for different sets of readers. I must, for some reason or other, have presupposed what Professor Bôcher explains. If my critic disapproves of this, it seems to me that he might have made clear the true trend of his objection, *which is irrelevant,* instead of making his readers believe that I had sinned against the first laws of logic, an intimation to which I cannot be indifferent. Professor Bôcher, who does not go far in the way of applications himself, might even have quoted my Apollonian paper as an illustration of the principles that underlie his doctrine; but, as a matter of fact, he meant to use it as a deterring example of their neglect, and only as such.

2. My fault (if fault it is) thus consisted in supposing notions widely known (and accepted by competent judges) which in reality perhaps were not so.* Why I held such an opinion is easily explained. The ideas in question are very simple. From the standpoint of Klein's Program of 1872 they appear almost as matters of course. To me they have been familiar since about 1884, the time when I was a young student under Professor Klein, who in later years was Professor Bôcher's teacher also, and to whom also Professor Bôcher, as he says himself, is indebted for the same ideas. It would seem to follow from this that the public I had in mind was likely to exist; and so it did, as is shown by the example of Professor Bôcher himself. But had it been nonexistent, I could still claim the right to write for it in the manner I did. What does not exist today may be common tomorrow. The first footnote in my paper shows that, notwithstanding its deplorable length, it is an extract only, and by no means intended for beginners.

3. I stated that I look upon the geometry of inversion as forming a *counterpart* (*Seitenstück*) *to projective geometry* (pages 498, 528), and I have treated it as such throughout. The idea apparently underlying Professor Bôcher's criticism,

* In my book of 1913 the point-continuum of the geometry of inversion is mentioned as a commonly accepted notion (page 282).

that in spite of this attitude (which was opposed to the habits of my predecessors, Darboux, for example) I might have operated in this projective domain seems quite absurd.

4. A piece of research in projective geometry is not necessarily defective if its author does not speak of points at infinity. The same holds good for the geometry of inversion.

5. That my point of view was *exactly* the one recommended by Professor Bôcher can be conclusively demonstrated. I say (page 539): "Um diese Aufgabe (das Apollonische Problem auf der Kugel) zu lösen, haben wir weder in Formeln noch Constructionen *auch nur die geringste Aenderung* anzubringen." Then there must be a one-to-one correspondence between the plane I operated in (the "plane of inversion") and the sphere, without any exceptional points.

Finally a remark may be made that has nothing to do with the subject at variance. The terms region and points *at infinity*, though very convenient, are apt to cause misunderstandings. It should be noticed that the (euclidean) distance between a point "at infinity" and an ordinary point is not necessarily infinite. Under certain conditions, the investigation of which I leave to the reader, it is indeterminate. The similar terms used in projective geometry require a similar comment.

I do not care to follow Professor Study into the field of personal recrimination. Whether my article was calculated to be of use is neither for him nor for me to say. What it aimed to accomplish is clearly set forth in its first twelve lines, and this aim would not have been affected by the insertion, demanded by him, of a reference to his book. This reference would, however, as a matter of course have been made if I had been acquainted with the passage. The very brief allusion to the paper on the Apollonian Problem is the merest incident in the course of my article. In spite of Professor Study's remarks, the autobiographic part of which seems to me irrelevant, I see no substantial alteration which should be made in my words. I never said or implied, thought or wished the reader to think that this paper is too long. I do think it of sufficient length to warrant the expectation that the author should state explicitly and exactly what he was talking about. Various inferences, correct or incorrect, might be drawn from his failure to do so. I drew none.

Maxime Bôcher.

THE DAVIS CALCULUS.

THE attack on our Calculus in the June BULLETIN perhaps calls for a word of protest. It is some consolation to know that the book has attractive elements, even though it is not stated what they are. Pleasant also is it that the reviewer recognizes that we did try to introduce interest into the subject. To get the student interested is certainly an important matter; and it is our belief that a text can and should help a teacher in doing so. The exciting of interest, however, is but a means to an end, that end being the stimulation of accurate thought on the many and varied applications of the calculus.

Mayhap the reviews of the book in the *American Mathematical Monthly*, in the *Bulletin* of the Society for Promoting Engineering Education, and in *Science* were far too favorable; yet, taken altogether, they would seem to indicate that we had been reasonably successful in carrying out our ideas. Still more satisfactory is it to know that of the seventy-odd institutions who will use the book for the coming year, a majority have already used it for three years.

Might we indeed possibly suggest that a review, to be valuable, should be accurate and judicial, not marred by exaggeration, by guesses as to how the book was written and what its reception will be, or by remarks on visionary ideals that have nothing to do with the use of the book?

ELLERY W. DAVIS.

NOTES.

THE July number (volume 16, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "Sur les fonctionnelles bilinéaires," by M. FRÉCHET; "Oriented circles in space," by D. F. BARROW; "A new isosceles triangle solution of the three body problem," by D. BUCHANAN; "Surfaces Ω and their transformations," by L. P. EISENHART; "The general theory of congruences," by E. J. WILCZYNSKI; "On matrices whose coefficients are functions of a single variable," by J. H. M. WEDDERBURN; "Conformal classification of analytic arcs or elements: Poincaré's local problem of conformal geometry," by E. KASNER; "Extensions of Descartes' rule of signs connected with a problem suggested by Laguerre," by D. R. CURTISS; "On parastrophic algebras," by J. B. SHAW.

THE July number (volume 37, number 3) of the *American Journal of Mathematics* contains: "Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface," by G. M. GREEN; "Linear combinants of systems of binary forms, with the syzygies of the second degree connecting them," by W. F. SHENTON; "Elimination d'une inconnue entre plusieurs équations algébriques," by M. STUYVAERT; "Congruences associated with a one-parameter family of curves," by R. D. BEETLE; "Plane sextic curves invariant under birational transformations," by A. HELEN TAPPAN.

THE academy of sciences of Paris has recently announced the following awards of prizes in the mathematical sciences:

Francoeur prize (fr. 1500) to Professor M. J. MARTY of the Lycée Alby, who has since been killed in battle.

Poncelet prize (fr. 2000) to Professor M. RABUT, emeritus professor in the Ecole des ponts et chaussées.

Boileau prize (fr. 1300) to Professor U. PUPPINI, of the University of Bologna.

AT the meeting of the Edinburgh mathematical society on June 11 the following papers were read: "On spheroidal harmonics and allied functions," by G. B. JEFFERY; "Determinants connected with the periodic solutions of Mathieu's equation," by A. G. BURGESS; "On the oscillation functions derived from a discontinuous function," by L. R. FORD; "The angle between two lines in trilinears," by W. L. MARR.

THE Helvetian society of natural scientists held its annual meeting at Geneva, September 12–15, this being its centenary celebration. The mathematical section was presided over by Professor H. FEHR.

A NEW edition of de Morgan's Budget of Paradoxes, edited by DAVID EUGENE SMITH with extended biographical, historical, and explanatory notes, has just been published by the Open Court Publishing Company.

The following advanced courses in mathematics are offered at the Italian universities during the academic year 1915–1916:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Classical problems of celestial mechanics, three hours.—By Professor L. DONATI: Modern electromagnetic theories. Thermodynamics and its relation to radiations; hypothesis of quanta, three hours.—By Professor F. ENRIQUES: Geometric theory of algebraic equations and functions, three hours.—By Professor S. PINCHERLE: Functional calculus. Integral equations and applications, three hours.

UNIVERSITY OF CATANIA.—By Professor E. DANIELE: Vibrations of elastic bodies. Theory of sound, four hours. —By Professor G. PENNACCHIETTI: Dynamics of rigid bodies (advanced part). Elasticity. Viscous fluids, four hours.—By Professor C. SEVERINI: Partial differential equations, four hours.—By ——: Higher geometry, three hours.

UNIVERSITY OF GENOA.—By Professor C. C. LEVI: Selected topics in the theory of partial differential equations, four and one half hours.—By Professor G. LORIA: Numerative geometry, three hours.—By Professor O. TEDONE: Electromagnetic theory of light, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics: the middle ages, three hours.—By Professor A. DEL RE: n-dimensional analysis of Grassmann, with application to mechanics of the spaces of constant curvature, four and one half hours.—By Professor R. MARCOLONGO: Fourier's series with several applications, three hours.—By Professor D. MONTESANO: Theory of birational correspondences of three-dimensional space, three hours.—By Professor E. PASCAL: Analytic functions and selected topics of mathematical analysis, three hours.—By Professor L. PINTO: Geometrical optics. Theory of optical instruments, three hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: Functions of a complex variable; elliptic functions; integral equations, four hours.—By Professor A. COMESSATTI: Projective and descriptive geometry of hyperspaces, three hours.—By Professor P. GAZZANIGA: Theory of numbers, three hours.—By Professor T. LEVI-CIVITA: Mechanics of continuous media: the technical, classical and relativistic point of view, four and

one half hours.—By Professor G. RICCI: Absolute differential calculus. General theory of elasticity, four hours.—By Professor F. SEVERI: Algebraic varieties from the point of view of reality, four hours.—By Professor A. SIGNORINI: Technical applications of elasticity, three hours.—By Professor A. TONOLO: Partial differential equations of the second order, three hours.—By Professor G. VERONESE: Geometrical applications of the theory of sets, four hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: Calculus of variations, three hours.—By Professor M. DE FRANCHIS: Hyperelliptic surfaces and Picard's varieties, three hours.—By Professor M. GEBBIA: Mechanics of continuous bodies. Potential. Hydrodynamics. External acoustics, four and one half hours.—By ———: Mechanics (advanced part), three hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Abelian integrals with geometrical applications mainly to correspondences between algebraic curves, three hours.—By Professor C. BOMPIANI: Geometry of numbers. Diophantine approximation, three hours.—By Professor U. CISOTTI: Electricity and magnetism, three hours.—By Professor F. GERBALDI: Elliptic functions, three hours.—By Professor G. VIVANTI: General theory of analytic functions, three hours.

UNIVERSITY OF PISA.—By Professor C. BERTINI: Geometry on a curve with algebraic and transcendental method, three hours.—By Professor L. BIANCHI: General theory of surfaces. Applicability. Rolling, four and one half hours.—By Professor U. DINI: Analytic representation of functions both in the real and in the complex field, four and one half hours.—By Professor G. A. MAGGI: Principles of analytic mechanics. Harmonic functions. Topics in hydrodynamics, four and one half hours.—By Professor P. PIZZETTI: Theory of the shape and of the motion of rotation of the planets, four and one half hours.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Geometrical applications of calculus, three hours.—By Professor G. CASTELNUOVO: Geometry of algebraic varieties, three hours.—By Professor U. CRUDELI: Introduction to the

advanced study of electricity, three hours.—By Professor V. Volterra: Electricity and magnetism, three hours. Differential, integro-differential and derivative-functional equations of mechanics, three hours.—By ———: Theory of functions of a complex variable. Elliptic functions, three hours.

University of Turin.—By Professor T. Boggio: Equilibrium-figures of rotating fluid masses, three hours.—By Professor G. Fubini: The modern advances of calculus. Application to expansions in series, to calculus of variations, to integral equations, three hours.—By Professor C. Segre: Topics in differential geometry, three hours.—By Professor C. Somigliana: Mechanical and electromagnetic optics, three hours.

Professor M. De Franchis, of the University of Catania, has accepted the professorship of higher geometry at the University of Palermo, as successor to the late Professor Guccia.

Professor E. Bertini, of the University of Pisa, has been elected national member of the royal academy of Turin.

Professors F. Enriques, of the University of Bologna, T. Levi-Civita, and F. Severi, of the University of Padua, have been elected corresponding members of the royal institute of Lombardy.

The Italian society of sciences (the forty) has awarded its gold medal for 1914, to Professor E. Pascal, of the University of Naples for his work concerning integrating factors of differential and of integro-differential equations.

Professor G. B. Mathews, of Bangor College, Wales, has received the honorary degree of doctor of laws from Cambridge University.

Dr. E. Hecke, of the University of Göttingen, has been appointed associate professor of mathematics at the University of Basel.

Dr. K. Knopp, of the University of Berlin, has been promoted to an associate professorship of mathematics.

PROFESSOR C. MOSER has been appointed professor of the theory of insurance at the University of Bern.

PROFESSOR F. SCHILLING, of the technical school at Dantzig, has received the title of Geheimer Regierungsrat.

PROFESSOR H. S. WHITE, of Vassar College, has received the degree of doctor of laws from Northwestern University.

PROFESSOR R. M. BARTON, of Lombard College, has been appointed dean and acting president.

DR. H. C. GOSSARD, of the University of Oklahoma, has been promoted to an assistant professorship of mathematics.

DR. W. L. MISER, of the University of Minnesota, has been appointed assistant professor of mathematics in the University of Arkansas.

DR. DANIEL BUCHANAN, of Queen's University, Kingston, Ontario, has been promoted to an associate professorship of mathematics and appointed director of the college observatory.

MR. J. L. RILEY has been appointed professor of mathematics in the Oklahoma state normal school at Tahlequah.

AT the State University of Iowa, Miss S. E. CRONIN has been promoted to an assistant professorship of mathematics.

PROFESSOR G. H. CRESSE, of Middlebury College, has been appointed assistant professor of mathematics in the University of Arizona.

DR. G. H. GRAVES, of Columbia University, has been appointed instructor in mathematics in Purdue University.

AT Princeton University Mr. L. S. HILL, of the University of Montana, has been appointed instructor in mathematics. Dr. H. GALAJIKIAN, has resigned his instructorship.

DR. H. B. PHILLIPS, of the Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematics.

Mr. T. Dantzig has been appointed instructor in mathematics at the University of Indiana.

Mr. C. E. Norwood has been appointed assistant in mathematics at Dartmouth College.

J. C. Wilson, professor of logic at Oxford University since 1889, died August 12, 1915. He was the author of "On Traversing Geometrical Figures," published by the Clarendon Press in 1905.

John Howard Van Amringe, emeritus professor of mathematics in Columbia University, died September 10, 1915, at the age of 80 years. He was an active member of the Columbia faculty for fifty years preceding his retirement in 1910. In 1894 he became dean of the college. He was a founder and the first president of the New York Mathematical Society, since reorganized as the American Mathematical Society.

John K. Sinclair, emeritus professor of mathematics in the Worcester polytechnic institute, died September 12, 1915.

Professor C. A. von Drach, of the University of Marburg, died recently at Cassel in his seventy-sixth year.

Professor O. Simony, of the agricultural institute at Vienna, died April 6, 1915.

Professor N. v. Sonin, minister of education and member of the Petrograd academy of sciences, died February 27, at the age of 66 years.

Professor A. Wernicke, of the technical school at Braunschweig, died March 30, at the age of 58 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Bagnera (G.). Lezioni di calcolo infinitesimale. Disp. 1–35. Palermo, tip. Matematica, 1914. 8vo. Pp. 1–280.

Ball (W. W. R.). Mathematical recreations and essays. 6th edition. New York, Macmillan, 1914. 8vo. 16+506 pp. $3.25

Balzini (A.). Il problema della quadratura del circolo risoluto. Pisa, A. Macchi, 1915. 4to. 7pp.+3 tavole.

Broggi (H.). Ecuaciones integrales lineales. La Plata, 1914. 36 pp.

Bützberger (F.). Ueber bizentrische Polygone, Steinersche Kreis- und Kugelreihen und die Erfindung der Inversion. 2ter Teil. Zurich, 1914. 8vo. 27pp.

Carl (A.). Zur Theorie der ebenen ähnlich-veränderlichen Systeme. (Diss.) Dresden, 1914.

Encyklopädie der mathematischen Wissenschaften. Band III 2, Heft 5: G. Loria, Spezielle ebene algebraische Kurven von höherer als der vierten Ordnung. Leipzig, Teubner, 1915. Gr. 8vo. Pp. 571–634. M. 2.40

Erler (H.). Metrische Relationen an den vollständigen Figuren und am Kegelschnitt. Breslau, 1914.

Fuss (H.). Modulsysteme und höhere komplexe kommutative Zahlsysteme. Kiel, 1913. 8vo. 70 pp.

Greco (D.). La quadratura del cerchio. Maglie, tip. Messapica di B. Canitano, 1915. 8vo. 15 pp.

Harms (F.). Die Transfiguration ebener Kurven von rechtwinkligen cartesischen auf A-normalen- und A-Strahlbüschelkoordinaten. Rostock, 1913. 8vo. 45 pp.+3 Tafeln.

Hintzer (F.). Die durch einen Kreispunkt gehenden geodätischen Linien auf dem elliptischen Paraboloid. Friedeberg, 1914. 4to. 23 pp.

Jentzsch (R.). Untersuchungen zur Theorie der Folgen analytischer Funktionen. (Diss.) Berlin, 1914.

Kissel (G.). Ueber den von den Trisektionslinien eines Dreiecks umhüllten Kegelschnitt. Giessen, 1913. 8vo. 50 pp.

Langhans (C.). Einführung in die Differentialrechnung. Plön, 1914. 8vo. 34 pp. M. 1.50

Loria (G.). See Encyklopädie.

MacMahon (P. A.). Combinatory analysis. Vol. I. Cambridge, University Press, 1915. 8vo. 19+300 pp. 15s.

Meyer (C.). Ueber den ersten Differentialquotienten des logarithmischen Kurvenpotentials. Hamm, 1913. 8vo. 37 pp.

Potthof (R.). Eine von Lagebeziehungen unabhängige geometrische Beweismethode und deren Anwendung. Wanne, 1914. 4to. 13 pp.

Rudolphi (W.). Analytische Geometrie des Raumes in Verbindung mit darstellender Geometrie. 3ter Teil: Kreiskegel und Kreiszylinder. Neumünster, 1914. 4to. 29 pp.+11 Tafeln.

STARKE (P.). Die Apollonischen Sehnenvierecke des Dreiecks. Leipzig, 1914. 4to. 28 pp. M. 1.60

WINKLER (R.). Ueber die Bewegung affin-veränderlicher ebener Systeme. (Diss., Techn. Hochschule Dresden.) Borna-Leipzig, R. Noske, 1914.

II. ELEMENTARY MATHEMATICS.

BAKER (A. E.). Commercial arithmetic. Chicago, Metropolitan Text Book Co., 1912. $1.00

BOHNERT (F.). Grundzüge der ebenen Geometrie. Leipzig, Göschen, 1915. 8vo. M. 2.80

BROWN (J. C.). Curricula in mathematics. (United States Bureau of Education Bulletin No. 619.) Washington, Government Printing Office, 1915. 91 pp. $0.10

CHADSEY (C. E.) and SKINNER (H. M.). Complete arithmetic. Boston, Atkinson and Mentzer, 1915. $0.65

HARPER (G. A.). See NEWELL (M. J.).

HJELMSLEV (J.). Geometrische Experimente. Aus dem Dänischen übersetzt von A. Rohrberg. Leipzig, Teubner, 1915. M. 2.40

LESSER (O.). See SCHWAB (K.).

LIETZMANN (W.). See SCHUSTER (M.).

LOHNERT (K.). Untersuchungen über die Auffassung von Rechtecken. Leipzig, 1913. 8vo. 78 pp.

LONGMANS explicit arithmetic. London, Longmans, 1915. 8vo. Pupils' Book V, cloth, 5d. Teachers' Book V, 1s.

LUMAN (J. A.). Practical comprehensive arithmetic. 5th edition. Philadelphia, Peirce School, 1914. $1.40

NEWELL (M. J.) and HARPER (G. A.). Plane and solid geometry. Chicago, Row and Peterson, 1915. $1.15

PERNT (M.). Ist das Rechnen nach Ferrol neu und vorteilhaft? Eine kritische Würdigung und eine Anleitung zum Rechnen mit Vorteil. Wien, J. Eberle, 1915. M. 0.60

REEVE (W. D.) and SCHORLING (R.). A review of high-school mathematics. Chicago, University of Chicago Press, 1915. 8vo. 10+70 pp. Cloth. $0.40

ROHRBERG (A.). See HJELMSLEV (J.).

SCHNEIDER (A.). See SCHWAB (K.).

SCHORLING (R.). See REEVE (W. D.).

SCHUSTER (M.). Geometrische Aufgaben und Lehrbuch der Geometrie nach konstruktiv-analytischer Methode, herausgegeben von W. Lietzmann. Ausgabe A, für Volkanstalten. 1ter Teil: Planimetrie. 4te Auflage. Leipzig, Teubner, 1914. 158 pp. Geb. M. 2.20

SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk zum Gebrauch an Lehrer- und Lehrerinnenbildungsanstalten. Im Sinne der Meraner Lehrpläne. 1ter Band: Lehr- und Uebungsbuch der Arithmetik und Algebra. Bearbeitet von A. Schneider. Leipzig, Freytag, 1915. M. 3.40

SCHWARTZ (A. J.). See YOUNG (J. W.).

SKINNER (H. M.). See CHADSEY (C. E.).

WALSEMANN (H.). Die Rechenkunst. Eine Anschauungs- und Denklehre der Zahl. 2te Auflage. Hannover–List, Meyer, 1913. 335 pp. M. 3.80

WEBSTER (J. H.). The Cambridge elementary arithmetics. Books I to VII. Cambridge, University Press, 1915. 3½d. to 6d. Teachers' Book IV, 1s. 6d.

YOUNG (J. W.) and SCHWARTZ (A. J.) Plane geometry. New York, Holt, 1915. $0.85

III. APPLIED MATHEMATICS.

ADLER (A. A.). The theory of engineering drawing. 2d edition, corrected. New York, Van Nostrand, 1915. 8vo. 12+315 pp. $2.00

AL-KHWARIZMI, MOHAMMED IBN MUSA. Die astronomischen Tafeln in der Bearbeitung des Maslama Ibn Ahmed Al-Madjzeti und der lateinischen Uebersetzung des Athelhard von Bath auf Grund der Vorarbeiten von A. Björnbo und R. Besthorn. Herausgegeben und kommentiert von H. Suter. Copenhagen, 1914. Lex. 8vo. 284 pp. M. 10.00

ASTRONOMISCHER Kalender 1915. Herausgegeben von der k. k. Sternwarte zu Wien. 3te Folge. 5ter Jahrgang. Wien, Gerold, 1915. 150 pp. Geb. M. 3.00

BAGNI (T.). Teoria matematica dei fenomeni collettivi. (Biblioteca del lavoro e degli affari per le scuole e per la vita.) Firenze, G. Barbèra (Alfani e Venturi), 1915. 16mo. 18+199 pp. L. 3.50

BARBIERI (A.). Poligonazione tacheometrica: norme pratiche per il rilevamento planimetrico delle linee poligonali e per il loro calcolo. (Manuali Hoepli.) Milano, Hoepli, 1915. 24mo. 11+246 pp. L. 2.50

BATEMAN (H). The mathematical analysis of electrical and optical wave-motion on the basis of Maxwell's equations. Cambridge, University Press, 1915. 8vo. 8+160 pp. 7s. 6d.

BENZI (N.). Dimostrazione matematica di un erroneo modo seguito per calcolare le semestralità. Acqui, S. Dina, 1915. 8vo. 15 pp.

BESTHORN (R.). See AL-KHWARIZMI.

BIRD (H. C.). Elements of descriptive geometry. Chester, Pa., Bird, 1914. $1.15

BJÖRNBO (A.). See AL-KHWARIZMI.

BOREL (E.). See ENCYCLOPÉDIE.

CHAUMONT (L.). Recherches expérimentales sur le phénomène électro-optique de Kerr et sur les méthodes servant à l'étude de la lumière polarisée elliptiquement. Paris, 1914. Gr. 8vo. 219 pp.

COSSERAT (E. and F.). See ENCYCLOPÉDIE.

DEBUS (H.). Die philosophischen Grundlagen des Relativitätsprinzips der Elektrodynamik. Bonn, 1913. 8vo. 55 pp. M. 1.80

DECKERT (A.). Infinitesimalrechnung mit Anwendung auf Naturwissenschaften und Technik. 1ter Teil: Differentialrechnung. Hildesheim, 1914. M. 6.00

EHRENFEST (P. and T.) See ENCYCLOPÉDIE.

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome IV, volume 1, fascicule 1: Principes de la mécanique rationnelle, exposé d'après l'article allemand de A. Voss par E. Cosserat et F. Cosserat; Mécanique statistique, exposé d'après l'article allemand de P. Ehrenfest et T. Ehrenfest par E. Borel. Leipzig, Teubner, 1915. Gr. 8vo. 6+290 pp. M. 11.00

FORTE (O.). Guida elementare alle esercitazioni di analisi chimica qualitativa. Napoli, G. Majo, 1915. 16mo. 76 pp. L. 2.00

GANS (R.). Estados correspondientes del magnetismo con una aplicación a la teoría del teléfono. La Plata, 1914.

GERRANS (H. T.). See MINCHIN (G. M.).

GOODMAN (J.). Mechanics applied to engineering. 8th edition. London and New York, Longmans, 1915. Cr. 8vo. 10+854 pp. $2.50

GROSSMANN (M.). Darstellende Geometrie. (Leitfaden für den mathematischen und technischen Hochschulunterricht.) Leipzig, Teubner, 1915. 8vo. 6+138 pp. Cloth. M. 2.80

HAGEMANN (W.). Ueber die Oberflächenspannung geschmolzener Metalle. Freiburg, 1914. 8vo. 48 pp.+4 Tafeln. M. 1.50

HART (I. B.). Elementary experimental statics. London, Dent, 1915. 8vo. 7+200 pp. 2s. 6d.

HISS (R.). Ueber die zeitliche Aenderung an Spannung reiner Flüssigkeitsoberflächen. Heidelberg, 1913. 8vo. 48 pp.

HOUSDEN (C. E.). New time savers in hydraulics and earthwork. London, Longmans, 1915. 3s.

HUGHES (A. L.). Die Lichtelektrizität. Deutsch von M. Iklé. Leipzig, 1915. M. 5.60

ICHAK (F.). Das Perpetuum mobile. (Aus Natur und Geisteswelt, Band 462.) Leipzig, Teubner, 1914. M. 1.25

IKLÉ (M.). See HUGHES (A. L.).

KÜHNE (E. E.). Definitive Bahnbestimmung des Kometen 1892 I (Swift) für die Oskulationsepoche 1892 März 21.0. Leipzig, 1913. 4to. 80 pp.

LEHMANN (P.). Die veränderlichen Tafeln des astronomischen und chronologischen Teils des Preussischen Normalkalenders für 1915. Berlin, 1915. Gr. 8vo. 5+135 pp. M. 6.80

LÖWE (S.). Ueber die erreichbare Genauigkeit der Widerstandsmessung in Hochfrequenzkreisen. Jena, 1913. 8vo. 60 pp. M. 2.00

MINCHIN (G. M.). A treatise on statics. Vol. II: Non-coplanar forces. 5th edition, revised by H. T. Gerrans. Oxford, Clarendon Press, 1915. 8vo. 8+369 pp. 10s. 6d.

MÜLLER (F. J.). Johann Georg von Soldner der Geodät. (Diss., Techn. Hochschule München.) München, 1914.

POSKE (F.). Didaktik des physikalischen Unterrichts. (Didaktische Handbücher für den realistischen Unterricht an höheren Schulen, Band 4.) Leipzig, Teubner, 1915. Gr. 8vo. Geb. M. 12.00

RUSSELL (A.). Treatise on the theory of alternating currents. Vol. I. 2d edition. Cambridge, University Press, 1915. 8vo. 14+534 pp. 15s.

SCHORR (R.). Die Hamburgische Sonnenfinsternis-Expedition nach Souk-Abras (Algerien) im August, 1905. 2ter Teil. Hamburg, 1914.

SLOCUM (S. E.). Elements of hydraulics. New York, McGraw-Hill, 1915. $2.50

SUTER (H.). See AL-KHWARIZMI.

TOWNSEND (J. S.). Electricity in gases. Oxford, Clarendon Press, 1915. 8vo. 16+496 pp. 14s

TUTTLE (L.). An introduction to laboratory physics. Philadelphia Jefferson Laboratory of Physics, 1915. 12mo. 150 pp. Cloth. $0.80

VIEWEGER (H.). Aufgaben und Lösungen aus der Gleich- und Wechselstromtechnik. Ein Uebungsbuch für den Unterricht an technischen Hoch- und Fachschulen, sowie zum Selbststudium. 4te verbesserte Auflage. Berlin, 1914. M. 7.00

VOSS (A.). See ENCYCLOPÉDIE.

WENNER (P.). Ein graphisches Ausgleichungsverfahren und dessen Anwendung auf astronomische Aufgaben. Heidelberg, 1913. 8vo. 39 pp.+2 Tafeln. M. 2.50

WEYRES (T.). Parallaxen von 8 Fixsternen abgeleitet mit dem unpersönlichen Uhrwerkmikrometer am kleinen Meridiankreis der Sternwarte zu Heidelberg. Heidelberg, 1914. 4to. 27 pp. M. 1.50

WITTENBAUER (F.). Aufgaben aus der technischen Mechanik. Band I. Allgemeiner Teil. 3te vermehrte und verbesserte Auflage. Berlin, 1914. M. 6.40

ON THE RELATION BETWEEN LINEAR ALGEBRAS AND CONTINUOUS GROUPS.

BY PROFESSOR L. E. DICKSON.

1. THE aim of this note is to give a very elementary account of the mutual relation between any linear associative algebra (system of hypercomplex numbers) and a type of continuous groups, without presupposing on the part of the reader a knowledge of either subject. The relation in question, first observed by Poincaré, enables us to translate the concepts and theorems of the one subject into the language of the other subject. It not only doubles our total knowledge, but gives us a better insight into either subject by exhibiting it from a new point of view. Incidentally, we shall obtain several other results of general interest.

2. To begin with the simplest illustration, we set up a correspondence between each real number c, not zero, and the transformation $z' = cz$, denoted by T_c, on the real variable z. The result of applying in succession T_c and the new transformation $T_{c'}$ (which we may express in the form $z'' = c'z'$) is the same as applying the single transformation $z'' = (c'c)z$. Hence we say that the *product* $T_cT_{c'}$ of the two given transformations is the transformation $T_{c''}$, where $c'' = c'c$. The set of transformations which correspond to the system (or algebra) of all real numbers, other than zero, is said to form a *group* G since the product of any two of these transformations is a transformation of the same set. In particular, G is a one-parameter continuous group. The relation $c'' = c'c$ between the parameters in $T_cT_{c'} = T_{c''}$ defines a transformation of c into c'' with the parameter c'. Since c' ranges over all real numbers other than zero, the resulting transformations $c'' = c'c$ on the parameters form a group which is the same as G, apart from the notation of the variables. Hence G is said to be its own parameter group.

Next, let z denote a complex variable $x + yi$ and let c range over all complex numbers $a + bi$ other than zero. Then T_c is equivalent to the binary transformation

$$T_{a,b}: \qquad x' = ax - by, \quad y' = bx + ay.$$

The set of transformations $T_{a,b}$ in which a and b range independently over all real numbers (with the exclusion of $a = b = 0$) forms a two-parameter real continuous group which is its own parameter group. While these facts can be readily verified by use of the binary transformations $T_{a,b}$ (and that method is recommended to the beginner as a desirable exercise), they follow at once from the earlier work, in which z is now interpreted to be a complex variable and c a complex parameter.

To the linear algebra of ordinary complex numbers $a + bi$, with real *coordinates* a, b and two *units* 1, i, therefore corresponds a two-parameter group of binary linear transformations $T_{a,b}$ in which the parameters a and b enter linearly and homogeneously, and such that the group is its own parameter group. Given, conversely, a group of this character, we can exhibit a corresponding linear associative algebra, the product of any two hypercomplex numbers c and z of which is the number z' such that the expanded form of the relation $z' = cz$ is that transformation of the group whose parameters are the coordinates of c. Additional simple illustrations of this statement are given in the following sections.

3. We shall obtain an important algebra by considering the linear transformations which leave unaltered the quadric surface S defined by

$$\begin{vmatrix} x_1 & x_2 \\ x_3 & x_4 \end{vmatrix} = 0.$$

The four variables and the coefficients of the transformations may be taken to be real numbers or to be ordinary complex numbers; either interpretation may be made by the reader, but the one chosen is to be retained throughout the discussion.

The surface S contains two sets of straight lines

$$L_k: \qquad x_1 = kx_3, \quad x_2 = kx_4,$$

$$\lambda_k: \qquad x_1 = kx_2, \quad x_3 = kx_4.$$

A linear transformation which replaces every plane through L_k by a plane through L_k is such that

$$x_1' - kx_3' = y_1(x_1 - kx_3) + y_3(x_2 - kx_4),$$

$$x_2' - kx_4' = y_2(x_1 - kx_3) + y_4(x_2 - kx_4),$$

in which $y_1, \cdots, y_4$ are linear functions of k. Let the trans-

formation leave unaltered three lines L_k. Then the preceding equations, quadratic in k, hold for three values of k and hence are identities in k. Since the left members are linear in k, we see that $y_1, \cdots, y_4$ are independent of k. Hence a linear transformation which leaves unaltered three lines L_k leaves unaltered every line L_k and is of the form

$$T_y:\quad \begin{aligned} x_1' &= y_1x_1 + y_3x_2, \quad & x_3' &= y_1x_3 + y_3x_4, \\ x_2' &= y_2x_1 + y_4x_2, \quad & x_4' &= y_2x_3 + y_4x_4, \end{aligned}$$

in which the parameters $y_1, \cdots, y_4$ are such that

$$y_1y_4 - y_2y_3 \neq 0.$$

If two transformations leave every L_k unaltered, their product leaves every L_k unaltered. Hence the set of all transformations T_y forms a group G. The direct verification of this fact will lead also to another needed property. The product $T_yT_{y'}$ is found to be $T_{y''}$, where

$$\begin{aligned} y_1'' &= y_1'y_1 + y_3'y_2, \quad & y_3'' &= y_1'y_3 + y_3'y_4, \\ y_2'' &= y_2'y_1 + y_4'y_2, \quad & y_4'' &= y_2'y_3 + y_4'y_4. \end{aligned}$$

These equations define a transformation with the parameters $y_1', \cdots, y_4'$ from the variables $y_1, \cdots, y_4$ to the variables $y_1'', \cdots, y_4''$; under this interpretation, the transformation is the same as $T_{y'}$, apart from the notation of the variables. Hence G is its own parameter group. According to the general statement at the end of § 2, the group G should correspond to a linear associative algebra. As the general element (or hypercomplex number*) of the algebra, we may take the matrix

$$x = \begin{Vmatrix} x_1 & x_2 \\ x_3 & x_4 \end{Vmatrix}.$$

The product xy is defined to be the matrix x' in which x_1', $\cdots$, x_4' are given by the equations marked T_y. Hence the group G defines the algebra whose elements are the matrices x; the general transformation T_y of G is merely the expanded form of the relation $x' = xy$ between matrices.

---

* For the exhibition of x as a linear combination of four units and the resulting linear aspect of the algebra, see the writer's Linear Algebras, Cambridge Tracts, 1914, pp. 3–5, p. 59.

Consider the product $\xi = yx$ of the same factors taken in reverse order. We obtain the transformation

$$T'_y: \quad \begin{array}{ll} \xi_1 = y_1x_1 + y_2x_3, & \xi_2 = y_1x_2 + y_2x_4, \\ \xi_3 = y_3x_1 + y_4x_3, & \xi_4 = y_3x_2 + y_4x_4. \end{array}$$

All such transformations form a group G'. This fact can be verified as above by forming the product $T'_yT'_{y'}$, or by showing that the T'_y leave unaltered every line λ_k and give all the linear transformations leaving unaltered every λ_k, or by the following third method. The product of T'_y, given by $\xi = yx$, and $T'_{y'}$, given by $\xi' = y'\xi$, is found by the elimination of ξ. Since $y'(yx) = (y'y)x$, the product is $T'_{y''}$, given by $\xi' = y''x$, where $y'' = y'y$.

Each transformation of G is commutative with each transformation of G' since $T_yT'_{y_1}$ is $\xi = y_1(xy)$, while $T'_{y_1}T_y$ is $\xi' = (y_1x)y$. The transformations of G and G' therefore generate the group Γ of the linear transformations $x' = y_1xy$. Suppose that this transformation is identical with $x' = Y_1xY$. Then $xA = Bx$ for every x, where $A = yY^{-1}$, $B = y_1^{-1}Y_1$. As is easily verified, the identity in x gives

$$A = B = \begin{Vmatrix} c & 0 \\ 0 & c \end{Vmatrix} \equiv S_c.$$

Then $Y = S_{c^{-1}}y$, $Y_1 = y_1S_c$. Hence Γ is a seven-parameter group.

To complete the discussion, we shall prove that the only linear transformations leaving the quadric surface S unaltered (i. e., automorphs of S) are the transformations of Γ (which permute the lines L_k among themselves and the lines λ_k among themselves) and their products by any one transformation, as (x_2x_3), which interchanges the two sets of lines. Let T be any linear automorph of S. If T replaces only a finite number of lines L_k by lines L_κ, it replaces an infinitude of lines L_k by lines λ_κ, so that the product of T by (x_2x_3) replaces an infinitude of lines L_k by lines L_κ. Hence either T itself or its product by (x_2x_3) is an automorph t which replaces an infinitude of lines L_k by lines L_κ. But we can find a transformation T'_y which replaces any three distinct lines L_k by any three distinct lines L_κ. This will evidently follow if we prove that there exists a transformation T'_y which replaces L_0, L_∞, L_1 by L_a, L_b, L_c, respectively, where a, b, c are any three distinct numbers; the conditions are

$$\frac{y_2}{y_4} = a, \quad \frac{y_1}{y_3} = b, \quad \frac{y_1 + y_2}{y_3 + y_4} = c,$$

and are satisfied when

$$y_4 = 1, \quad y_2 = a, \quad y_3 = \frac{c - a}{b - c}, \quad y_1 = by_3.$$

The product of t by the inverse of the first T_y' leaves unaltered three lines L_k and hence is a transformation T_y, as proved above. Thus t is in the group Γ and hence either T is in Γ or T is the product of a transformation of Γ by (x_2x_3).

The group of all linear automorphs of S is therefore of the kind called a mixed group. It is however determined in a very simple manner from the continuous seven-parameter subgroup Γ composed of the transformations $x' = y_1xy$. As in this instance, the introduction of hypercomplex numbers enables us to give a very compact and convenient notation for the transformations of important groups.

4. From the preceding algebra whose elements are matrices we can derive in a very natural manner the algebra of quaternions and deduce as corollaries several important results. To this end we take the interpretation which assigns ordinary complex values to the variables and coefficients of the transformations in § 3. To transform the equation of the quadric surface into

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 = 0,$$

we have merely to write

$$x_1 = X_1 + iX_4, \quad x_4 = X_1 - iX_4, \quad x_2 = -X_2 + iX_3,$$
$$x_3 = X_2 + iX_3.$$

The new form of transformation T_y involves the parameters only in the combinations $y_4 \pm y_1$, $y_2 \pm y_3$. Hence we write

$$y_4 + y_1 = 2Y_4, \quad y_4 - y_1 = 2iY_1, \quad y_2 - y_3 = 2Y_3,$$
$$y_2 + y_3 = 2iY_2.$$

In terms of the new variables and parameters, T_y becomes

$$t_Y: \quad \begin{aligned} X_1' &= X_4Y_1 - X_3Y_2 + X_2Y_3 + X_1Y_4, \\ X_2' &= X_3Y_1 + X_4Y_2 - X_1Y_3 + X_2Y_4, \\ X_3' &= -X_2Y_1 + X_1Y_2 + X_4Y_3 + X_3Y_4, \\ X_4' &= -X_1Y_1 - X_2Y_2 - X_3Y_3 + X_4Y_4. \end{aligned}$$

The identical transformation $X_1' = X_1$, etc., is obtained by taking $Y_1 = Y_2 = Y_3 = 0$, $Y_4 = 1$. To our group therefore corresponds a linear associative algebra whose general number is

$$X = X_1 i + X_2 j + X_3 k + X_4,$$

where the products of the units $i, j, k, 1$ are such that

$$XY = X' \equiv X_1' i + X_2' j + X_3' k + X_4',$$

in which the values of $X_1', \cdots, X_4'$ are given by t_Y. Taking $X_1 = Y_1 = 1$, $X_s = Y_s = 0$ $(s = 2, 3, 4)$, we find that $ii = -1$. In this way, we get

$$Q: \quad \begin{gathered} i^2 = j^2 = k^2 = -1, \quad ij = k, \quad ji = -k, \\ jk = i, \quad kj = -i, \quad ki = j, \quad ik = -j. \end{gathered}$$

We thus obtain the algebra of quaternions with ordinary complex coordinates. In view of its origin it is equivalent under a linear transformation on the units to the algebra of matrices with complex coordinates (§ 3). It has as a subalgebra the system of real quaternions.

The transformation t_Y leaves the quadric surface unaltered. By finding the coefficient of X_1^2, we see that

$$\sum_{s=1}^{4} X_s'^2 = \sum_{s=1}^{4} Y_s^2 \cdot \sum_{s=1}^{4} X_s^2.$$

The left member is called the *norm* of the quaternion X'. Since the transformation is $X' = XY$, we conclude that the norm of the product of two quaternions equals the product of their norms.

By interchanging X_s and Y_s $(s = 1, 2, 3, 4)$ in t_Y, we obtain the transformation t_Y' which has the more compact notation $X' = YX$. The product of the commutative transformations t_Y and t_{Y_1}' is $X' = Y_1 X Y$. The latter form a seven-parameter continuous group Γ. The determinant of each of its real transformations is positive, since the determinant of T_y in § 3 is the square of

$$y_1 y_4 - y_2 y_3 = Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2.$$

To $(x_2 x_3)$ corresponds the transformation τ which changes the sign of X_2 without altering X_1, X_3, X_4. The only linear automorphs of the surface are the transformations of Γ and

their products by τ (§ 3), these products having negative determinants. In four-dimensional space these products are reflexions, so that the group generated by the rotations around the origin and the stretchings from it is formed of the transformations $q' = q_1 q q_2$, where q and q' are variable real quaternions, while q_1 and q_2 are real quaternion parameters. Concerning this group, the corresponding one in three dimensons, and references on related subjects, see Linear Algebras, page 61.

5. Our final illustration will be more typical of the general theory since it treats a group not initially its own parameter group. Consider any two-parameter binary linear group in which the parameters Y_1, Y_2 enter linearly and homogeneously. Its transformations are therefore of the form

$$x_1' = (AY_1 + BY_2)x_1 + (CY_1 + DY_2)x_2,$$

$$x_2' = (EY_1 + FY_2)x_1 + (GY_1 + HY_2)x_2.$$

Let $Y_1 = a$, $Y_2 = b \neq 0$ be the values of the parameters giving the identical transformation. Introduce the new parameters

$$y_1 = bY_1 - aY_2, \quad y_2 = Y_2/b.$$

Then the values $y_1 = 0$, $y_2 = 1$ give the identical transformation. The new equations of our transformations will be of the above form, in which now $B = H = 1, D = F = 0$. Further, we set $AY_1 + Y_2 = y_2$, $Y_1 = y_1$. Hence the transformation becomes

$$T_y: \quad x_1' = y_2 x_1 + c y_1 x_2, \quad x_2' = a y_1 x_1 + (d y_1 + y_2) x_2.$$

The product $T_y T_{y'}$ is seen to be $T_{y''}$, where

$$P: \; y_1'' = (y_2' + d y_1') y_1 + y_1' y_2, \quad y_2'' = a c y_1' y_1 + y_2' y_2.$$

Hence the totality of transformations T_y forms a group G. Regarding y_1' and y_2' as the parameters, we have the general transformation of the parameter group of G. Thus G is its own parameter group only when $d = 0$, $c = 1$.

Without loss of generality we may take $c = 1$. If $c \neq 0$, this may be done by taking cy_1 as a new y_1. If $c = 0, a \neq 0$, we interchange x_1 and x_2 and take $dy_1 + y_2$ as a new y_2, obtaining T_y with $c \neq 0$. If $c = a = 0$, the case $d = 0$ is excluded since the group has two parameters, so that $dy_1 + y_2$ may be taken as the new y_1. Then

$$g: \quad x_1' = y_2 x_1, \quad x_2' = y_1 x_2.$$

Using the new variables $X_1 = x_1 + x_2$, $X_2 = x_1 - x_2$ and new parameters $Y_1 = (y_2 - y_1)/2$, $Y_2 = (y_2 + y_1)/2$, we get

$$g': \qquad X_1' = Y_2 X_1 + Y_1 X_2, \quad X_2' = Y_1 X_1 + Y_2 X_2,$$

which is of type T_Y with $c = a = 1$, $d = 0$.

There is a general method of selecting new variables X_1 and X_2* such that the group on the new variables will become its own parameter group. In the equations for T_y we have only to erase the accents in the left members, replace y_1, y_2 by X_1, X_2 and give to x_1, x_2 such special values that the resulting equations are independent. We may take $x_1 = 0$, $x_2 = 1$, and get

$$x_1 = X_1, \quad x_2 = dX_1 + X_2.$$

Expressed in the new variables, the transformation T_y (with $c = 1$) becomes

$$t_y: \quad X_1' = (y_2 + dy_1)X_1 + y_1 X_2, \quad X_2' = ay_1 X_1 + y_2 X_2.$$

In view of P, the group of these transformations is its own parameter group. For $y_1 = 0$, $y_2 = 1$, the transformation is identity. Hence we obtain an algebra with units e, ϵ, where ϵ is the principal unit, such that, if $y = y_1 e + y_2 \epsilon$ is its general number, $Xy = X'$, where the coordinates of X' are defined by t_y. The multiplication table is therefore

$$\epsilon^2 = \epsilon, \quad \epsilon e = e\epsilon = e, \quad e^2 = de + a\epsilon.$$

Taking $e - \frac{1}{2}d\epsilon$ as a new e, we have $d = 0$. Then multiplying e by r, we see that a is replaced by $r^2 a$, which may be made equal to 0 or 1 by choice of r. Hence there are just two types of binary algebras with complex coordinates and having a principal unit.

The corresponding groups are composed of the transformations t_y, with $d = 0$, $a = 0$ or 1. That with $a = 1$ is g' and was seen to be equivalent to g. Hence every binary linear group in which the two parameters enter linearly and homogeneously is equivalent to g or to the group h of transformations t_y with $a = d = 0$.

Scheffers proceeded in the reverse order. Making use of Lie's determination of all types of binary linear groups, he selected the two-parameter groups in which the parameters

* Lie-Scheffers, Continuierliche Gruppen, 1893, p. 634.

enter linearly and homogeneously, found their finite equations, and introduced variables such that each group becomes its own parameter group. The resulting groups (l. c., page 648, bottom, and page 649) are our h and g. From these he derived the above two algebras.

6. Scheffers' determination (pages 654–6) of the algebra of quaternions is based upon the existence of the group of transformations t'_Y of § 4. In a rather arbitrary manner he selected four infinitesimal transformations out of an aggregate of the ∞^6 infinitesimal automorphs of the quadric surface, and verified that the four generate a four-parameter group. The guide to this seemingly fortunate selection may well have been the previous knowledge of the group defined by the algebra of quaternions. The above discussion in § 4 not only gives a natural derivation of quaternions from the theory of groups but leads to the total group of automorphs of a quadric surface and not merely to its continuous subgroup.

THE UNIVERSITY OF CHICAGO.

AN ASPECT OF THE LINEAR CONGRUENCE WITH APPLICATIONS TO THE THEORY OF FERMAT'S QUOTIENT.

BY MR. H. S. VANDIVER.

(Read before the American Mathematical Society, August 4, 1915.)

IN 1903, Professor G. D. Birkhoff communicated to me the following theorem:

If p is a prime integer and a is a positive integer prime to p, then there is at least one and not more than two sets (x, y) such that

$$a \equiv \pm x/y \pmod{p}$$

where x and y are integers prime to each other and $0 < x < \sqrt{p}$, $0 < y < \sqrt{p}$.

Professor Birkhoff has kindly allowed me to use this result, and in the present paper I shall give a proof of the theorem which involves a continued fraction algorithm for a direct determination of each set. Some extensions and applications are also given.

1. *Proof and Algorithm.*—We have

$$p = am_1 + r_1 \quad (0 < r_1 < a),$$

$$a = m_2r_1 + r_2 \quad (0 < r_2 < r_1),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$r_{k-2} = m_kr_{k-1} + r_k \quad (0 < r_k < r_{k-1}),$$

$$r_k < \sqrt{p} \leqq r_{k-1}.$$

These relations give

$$(m_km_{k-1} + 1)r_{k-2} = r_k + m_kr_{k-3}$$

and similarly

$$r_{k-3}(m_km_{k-1}m_{k-2} + m_k + m_{k-2}) = - r_k + r_{k-4}(m_km_{k-1} + 1).$$

Hence we ultimately have

$$al \equiv \pm r_k \pmod{p},$$

where l is the continuant

$$\begin{vmatrix} m_k & 1 & 0 & \cdots & & 0 \\ -1 & m_{k-1} & 1 & \cdots & & 0 \\ 0 & -1 & m_{k-2} & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -1 & m_1 \end{vmatrix}.$$

Note that l is prime to p since r_k is so.

We now prove that $l < \sqrt{p}$. We have

$$r_{k-3} = (m_{k-1}m_k + 1)r_{k-1} + m_{k-1}r_k,$$

$$r_{k-4} = (m_km_{k-1}m_{k-2} + m_k + m_{k-2})r_{k-1} + r_k(m_{k-1}m_{k-2} + 1),$$

and finally

$$p = lr_{k-1} + r_kj, \tag{1}$$

where j equals

$$\begin{vmatrix} m_{k-1} & 1 & 0 & \cdots & & 0 \\ -1 & m_{k-2} & 1 & \cdots & & 0 \\ 0 & -1 & m_{k-3} & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -1 & m_1 \end{vmatrix}.$$

Hence, since $r_{k-1} \geqq \sqrt{p}$ then $l < \sqrt{p}$ because $r_k j > 0$. Also r_k is prime to l since l is prime to p. Therefore one of the sets (x, y) is (r_k, l). Set $x_0 = r_k$, $y_0 = l$. Then if there were a second set (x_1, y_1) where x_1 is prime to y_1, we should have

$$x_0 y_1 + x_1 y_0 = p,$$

since x_0 and y_0 are prime to each other. This being the case, we see from (1), after noting that $r_{k-1} \geqq \sqrt{p}$, that there must exist a positive integer μ such that

$$y_0(r_{k-1} - \mu x_0) + x_0(j + \mu y_0) = p,$$

where

$$r_{k-1} - \mu x_0 < \sqrt{p},\ j + \mu y_0 < \sqrt{p}.$$

These two conditions may be written

$$\frac{r_{k-1} - \sqrt{p}}{x_0} < \mu < \frac{\sqrt{p} - j}{y_0}. \tag{1a}$$

Now μ cannot have more than one integral value, since the above relation would then give

$$-p + (x_0 + y_0)\sqrt{p} > 2x_0 y_0, \tag{2}$$

which shows that one of the integers x_0, y_0, is $> \frac{1}{2}\sqrt{p}$. If $y_0 > \frac{1}{2}\sqrt{p}$ and $x_0 = 1$, then (2) evidently does not hold. Also, if $x_0 > 1$, the right-hand member of (2) increases faster than the left-hand member when x_0 is increased. Since, then, μ has no more than one positive integral value, there are not more than two sets (x, y) satisfying the conditions of the theorem. Further, μ, when it exists, is uniquely determined by (1*a*). This completes the proof and algorithm.

The existence of one set (x, y) may also be shown by means of a theorem due to Minkowski.*

If

$$\begin{aligned} f_1 &= a_{11} u_1 + \cdots + a_{1m} u_m, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f_m &= a_{m1} u_1 + \cdots + a_{mm} u_m \end{aligned}$$

are m linear homogeneous forms in $u_1, u_2, \cdots, u_m$ with arbitrary real coefficients $a_{11}, \cdots, a_{mm}$ of determinant Δ, then it is

* Geometrie der Zahlen, page 104.

always possible to select integers for $u_1, u_2, \cdots, u_m$ so that

$$|f_i| \leqq \sqrt[m]{\Delta} \qquad (i = 1, 2, \cdots, m).$$

In this relation set all the $a_{ij} = 0$ for $i > 1$, except when $i = j$ in which case $a_{ii} = 1$, $i = 2, 3, \cdots, m$. Then

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = a_{11}.$$

Assume further that $a_{11}, a_{12}, \cdots, a_{1m}$ are integers, then the result may be expressed as follows:

It is always possible to choose integers $u_2, u_3, \cdots, u_m$ such that

$$f_1 = a_{12}u_2 + \cdots + a_{1m}u_m \pmod{M},$$

where $a_{12}, \cdots, a_{1m}$ and M are given integers and $|f_1| \leqq \sqrt[m]{M}$, $|u_i| \leqq \sqrt[m]{M}$ $(i = 2, 3, \cdots, m)$. If we further put $m = 2$, we obtain

$$a_{12}u_2 \equiv f_1 \pmod{M},$$

where $|f_1| < \sqrt{p}$ and $|a_{12}| < \sqrt{p}$. We note that μ is not necessarily prime. The existence of at least one set (a_{12}, f_1) for a composite modulus M may also be shown by the algorism which has just been explained for the derivation of the first set (x_0, y_0) in

$$ay \equiv \pm x \pmod{p}.$$

Evidently the reasoning used also holds for the case where p is composite.

2. *Applications to Fermat's Quotient.*—The congruence

$$x^{p-1} \equiv 1 \pmod{p^2}, \tag{3}$$

where p is an odd prime, has $p - 1$ incongruent roots modulo p^2. By the theorem just proved, each can be represented by an expression of the type $\pm\, m/n$, where m and n are positive integers each $< p$. If

$$\left(\pm \frac{m}{n}\right)^{p-1} \equiv 1 \pmod{p^2},$$

then

$$m^{p-1} \equiv n^{p-1} \pmod{p^2}.$$

The relation (3) has the roots ± 1. If it has another root, say $\pm m_{10}/m_{20}$, when numerator and denominator are each positive and $< p$, then assume that there are k positive integers $m_{k0} < p$ such that

$$m_{10}{}^{p-1} \equiv m_{20}{}^{p-1} \equiv \cdots m_{k0}{}^{p-1}$$

modulo p^2, when $m_{a0} \mp m_{b0}$. Then we may form the $2k(k-1)$ expressions

$$\pm \frac{m_{10}}{m_{20}}, \pm \frac{m_{10}}{m_{30}}, \cdots \pm \frac{m_{30}}{m_{20}}, \cdots,$$

$$\pm \frac{m_{20}}{m_{10}}, \pm \frac{m_{30}}{m_{10}}, \cdots \pm \frac{m_{20}}{m_{30}}, \cdots,$$

each of which satisfies (3). If the $p-1$ roots of (3) exclusive of ± 1, are not exhausted by this set, then in like manner, there must exist m's such that

$$m_{11}{}^{p-1} \equiv m_{21}{}^{p-1} \equiv \cdots \equiv m_{k_1 1}{}^{p-1},$$

where

$$m_{a1} \mp m_{b1} \text{ and } m_{c1} \mp m_{d0} \text{ and } k_1 \geqq 2.$$

As before, we may form a set of solutions of (3) by means of these m's in the same way the former solutions were set up. If this second set does not exhaust the remaining roots of (3), the process may be repeated. Ultimately we obtain a set

$$\pm \frac{m_{1i}}{m_{2i}}, \pm \frac{m_{1i}}{m_{3i}}, \cdots, \pm \frac{m_{3i}}{m_{2i}}, \cdots,$$

$$\pm \frac{m_{2i}}{m_{1i}}, \pm \frac{m_{3i}}{m_{1i}}, \cdots, \pm \frac{m_{2i}}{m_{3i}}, \cdots,$$

where

$$m_{1i}{}^{p-1} \equiv m_{2i}{}^{p-1} \equiv \cdots \equiv m_{k_i i}{}^{p-1},$$

such that it includes, when taken together with the preceding i sets and the special roots ± 1, all the incongruent roots of (3). We have evidently

$$\sum_{j=0}^{i} 2k_j(k_j - 1) \geqq p - 3.$$

Assume $i = 0$, then

$$2k_0^2 - 2k_0 \geqq p - 3,$$

whence

$$k_0 \geqq \frac{1 + \sqrt{2p - 5}}{2}.$$

Consider the set

$$1^{p-1}, \quad 2^{p-1}, \quad 3^{p-1}, \cdots, \quad (p-1)^{p-1},$$

and call the least positive residues of these integers modulo p^2, *proper residues* modulo p^2. If $i = 0$, then from

$$m_{10}^{p-1} \equiv m_{20}^{p-1} \equiv \cdots \equiv m_{k0}^{p-1}$$

we are enabled to conclude that there are not more than

$$p - \frac{1 + \sqrt{2p - 5}}{2}$$

incongruent proper residues modulo p^2. Assume now that $i = 1$. We have

$$2k_1^2 + 2k_0^2 - 2k_1 - 2k_0 \geqq p - 3,$$

whence

$$k_1 + k_0 - 2 \geqq \frac{1 + \sqrt{2p - 5}}{2} - 1.$$

Hence we conclude as before that there are not more than

$$p - \frac{1 + \sqrt{2p - 5}}{2}$$

incongruent proper residues modulo p^2. In general we have

$$\sum_{j=0}^{i} k_j - (i + 1) \geqq \frac{1 + \sqrt{2p - 5}}{2} - 1.$$

Now consider a lower limit for the number of incongruent proper residues modulo p^2. There cannot be less than $[\sqrt{p}]$ residues of this type, for if there were only $[\sqrt{p}] - 1$ residues then there must exist at least one set of distinct positive integers $m_i < p$ such that

$$m_1^{p-1} \equiv m_2^{p-1} \equiv \cdots \equiv m_s^{p-1} \pmod{p^2},$$

where $s > [\sqrt{p}]$. This being the case, consider

$$(p - m_i)^{p-1}, \qquad (i = 1, 2, \cdots, s).$$

Each of these gives a proper residue and they are all incongruent modulo p^2. Evidently, also, there are at least $[\sqrt{p}]$ of them, contrary to hypothesis. Hence the theorem:

There are not more than

$$p - \frac{1 + \sqrt{2p - 5}}{2}$$

and not less than $[\sqrt{p}]$ *incongruent proper residues modulo* p^2, *where* p *is prime* > 2.

3. We now consider the relation of the foregoing to Fermat's last theorem. If

$$x^p + y^p + z^p = 0$$

is satisfied in integers prime to each other and to the odd prime p, then $2^{p-1} \equiv 1 \pmod{p^2}$, a result due to Wieferich. This being the case, then the set

$$1^{p-1}, \; 3^{p-1}, \cdots, (p - 2)^{p-1}$$

evidently includes all the incongruent proper residues modulo p^2. Hence they are not more than $\frac{1}{2}(p - 1)$ in number. Similarly, using the criterion of Mirimanoff, $3^{p-1} \equiv 1 \pmod{p^2}$, we note that the forms

$$(1 + 6k)^{p-1}, \quad (5 + 6k_1)^{p-1}$$

include all the incongruent proper residues and there are not more than $2[p/6]$. We may further reduce the number by using the criteria $5^{p-1} \equiv 11^{p-1} \equiv 17^{p-1} \equiv 1 \pmod{p^2}$.*

* Vandiver, *Journal für Mathematik*, vol. 144, p. 314. Frobenius, *Sitzungsberichte der K. Akademie der Wissenschaften*, Berlin, 1914, p. 653.

LIMITS OF THE DEGREE OF TRANSITIVITY OF SUBSTITUTION GROUPS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, August 3, 1915.)

THE main object of the present paper is to establish an elementary theorem which gives always a smaller upper limit for the degree of transitivity of a substitution group of degree $n > 12$ which does not include the alternating group of this degree, than the one given by the commonly quoted theorem that this limit cannot exceed $\frac{1}{3}n + 1$.* The theorem to be established is a generalization of the one published by the present writer in volume 4, page 140, of this BULLETIN. In Pascal's Repertorium, loc. cit., a footnote states that the limit $\frac{1}{3}n + 1$ is actually attained by the five-fold transitive Mathieu group of degree 12. In view of the results of the present paper this footnote could be completed by adding that this limit cannot be attained for any degree which exceeds 12. It is clearly also attained when $n = 6$, although this is not mentioned in the footnote.

Let G be any transitive substitution group of degree $n = kp + r$, where p is a prime number such that $p > k$, and $r > k$, and all the symbols p, r, k represent positive integers. In what follows it will always be assumed that G is neither alternating nor symmetric on the n letters and that $k > 1$. If G is more than r-fold transitive it includes a transitive subgroup H of degree kp, and hence its order is divisible by p. A Sylow subgroup of order p^α contained in H must be intransitive and each of its transitive constituents must be of degree p, since G cannot involve a substitution composed of a single cycle of degree p, according to the well-known theorem that a primitive group which involves a cyclic substitution of degree p cannot be of degree greater than $p + 2$ unless it includes the alternating group of its own degree.

It may be assumed that H is composed of all the substitutions of G on a certain set of kp letters. Since it is assumed that

* Cf. Pascal's Repertorium der höheren Mathematik, vol. 1 (1910), p. 211; Encyclopédie des Sciences mathématiques, tome 1, vol. 1, p. 549; etc.

G is at least r-fold transitive there are at least $r!$ substitutions in G which transform among themselves the letters not contained in H according to the symmetric group on these r letters. In fact, the exact number of these substitutions is the order of H multiplied by $r!$, and all of these substitutions constitute a group H' which involves H invariantly. The Sylow subgroups of order p^α contained in H form a complete set of conjugates under H', and hence each of these Sylow subgroups is transformed into itself under H' by $r!$ times as many substitutions as under H. The largest subgroup of H' which transforms one of these Sylow subgroups into itself must have for one of its transitive constituents the symmetric group on the r letters of G which are not contained in H. We proceed to prove that this is impossible and hence that the assumption that G is more than r-fold transitive leads to an absurdity.

The Sylow subgroups of order p^α must be of degree kp, since H is transitive and its degree is divisible by p. Let P represent one of these subgroups and consider the group P' formed by all the substitutions of H' which transform the abelian group P into itself. The subgroup of P' which is composed of all the substitutions of P' which do not interchange any of the systems of intransitivity of P must be invariant. This subgroup P_1 must include P and the quotient group P_1/P must be cyclic, as we proceed to prove. It is at once evident that this quotient group is abelian, since the group of isomorphisms of the group of order p is cyclic and the transitive constituents of P_1 are all of degree p. Hence we may assume that the systems of intransitivity of P are transformed under P' according to the symmetric group of degree k. If the substitutions of P_1 did not transform into itself every subgroup of order p contained in P, it would follow that P could not contain a substitution involving a minimum number of cycles when this number is greater than unity.

When $r > 4$ it is clearly not necessary to prove that P_1/P is cyclic, since the alternating group whose degree exceeds 4 is simple. As the symmetric group of degree r constitutes a transitive constituent of P' and as the systems of intransitivity of P are transformed under P' according to a group whose order cannot exceed $k!$, it results that the part of P' which corresponds to the alternating group on $r!$ letters is the direct product of H and this alternating group. As this is impossible since G does not include the alternating group of degree n,

we have established the following theorem: *A group of degree* $n = kp + r$, $p > k$ *and* $r > k$, *which is not alternating or symmetric, cannot be more than* r *times transitive unless* $k = 1$ *and* $r = 2$.

To prove that this theorem gives a much smaller upper limit for the degree of transitivity than $\frac{1}{3}n + 1$ whenever n is large, it is only necessary to use the well-known postulate of J. Bertrand, first proved by P. L. Tchebychef, that there is at least one prime number between x (exclusive) and $2x - 2$ (inclusive), whenever $x \geqq 3\frac{1}{2}$. Hence there is at least one prime number between $\sqrt{n}$ and $2\sqrt{n} - 2$ when $n > 12$, since n is an integer in the present consideration. If n is divided by this prime the quotient k is less than $\sqrt{n}$ and the remainder r must be less than $2\sqrt{n} - 2$. If this remainder does not exceed k we diminish k by unity and thus get a value of r which is less than $3\sqrt{n} - 2$ and greater than k. Hence it results from the theorem above that *when* $n > 12$ *a group of degree* n *cannot be* $(3\sqrt{n} - 2)$*-fold transitive.*

When $n \geqq 100$ this clearly gives a smaller upper limit for the degree of transitivity than $\frac{1}{3}n + 1$. That the theorem also gives a smaller upper limit when n lies between 12 and 100 can be easily verified directly. In fact, according to this theorem a group of degree 13 which is neither alternating nor symmetric cannot be more than triply transitive since $13 = 2\cdot 5 + 3$. Such a group of degree 14 cannot be more than triply transitive since $14 = 11 + 3$, and hence such a group of degree 15 cannot be more than four-fold transitive. Such groups of degrees 16 and 17 cannot be more than triply transitive since $16 = 13 + 3$ and $17 = 2\cdot 7 + 3$, and hence such a group of degree 18 cannot be more than four-fold transitive. Such a group of degree 19 cannot be more than four-fold transitive since $19 = 3\cdot 5 + 4$. Such groups of degrees 20 and 22 cannot be more than triply transitive since $20 = 17 + 3$ and $22 = 19 + 3$, and hence such groups of degrees 21 and 23 cannot be more than four-fold transitive.

Similar considerations readily lead to the result that a group whose degree is less than 159, and which does not include the alternating group of its degree, cannot be as much as 8-fold transitive. In fact, by means of a table of prime numbers, it is very easy to verify that such a group can not be as much as 15-fold transitive, according to the theorem above, unless

its degree exceeds 1,000, while the formula $\frac{1}{3}n + 1$ would place the upper limit of transitivity for such groups beyond 300. These illustrations may suffice to exhibit clearly that a much smaller upper limit for the degree of transitivity of a primitive group which is neither alternating nor symmetric results from the use of the present theorem than the one given by $\frac{1}{3}n + 1$, whenever n is large. When $n = 12 = 7 + 5$ the two theorems lead to the same upper limit. This is also true for the cases when n is 8 or 9. Since the groups whose degrees are less than 8 are so well known, it does not appear necessary to preserve the formula $\frac{1}{3}n + 1$ as an upper limit of the degree of transitivity of substitution groups which do not include the alternating group, especially since the theorem proved above is based upon such very elementary considerations.

University of Illinois.

THE PERMUTATIONS OF THE NATURAL NUMBERS CAN NOT BE WELL ORDERED.

BY PROFESSOR A. B. FRIZELL.

(Read before the American Mathematical Society, February 27, 1915.)

Let us tabulate the natural numbers according to the number of their prime factors, viz., the nth row shall consist of the products $\pi(\nu, n)$ of n primes in order of magnitude. Form a new rectangular array wherein the nth column shall be composed of numbers from the nth row of the first scheme but arranged in rows by their column indices ν in the former, so that now the ith row contains those products $\pi(\nu, n)$ for which ν is a product of i primes. We obtain an infinite matrix of series

$$\begin{array}{l}
3,\ 5,\ 11,\ 17,\ 31,\ \cdots;\ 6,\ 9,\ 14,\ 21,\ 33,\ \cdots;\\
\quad 12,\ 18,\ 27,\ 30,\ 50,\ \cdots;\ 24,\ 36,\ 54,\ 60,\ 90,\ \cdots;\ \cdots\\
7,\ 13,\ 23,\ 29,\ 43,\ \cdots;\ 10,\ 15,\ 25,\ 26,\ 38,\ \cdots;\\
\quad 20,\ 28,\ 44,\ 45,\ 66,\ \cdots;\ 40,\ 56,\ 84,\ 88,\ 126,\ \cdots;\ \cdots\\
19,\ 37,\ 61,\ 71,\ 103,\ \cdots;\ 22,\ 34,\ 51,\ 57,\ 82,\ \cdots;\\
\quad 42,\ 52,\ 76,\ 92,\ 116,\ \cdots;\ 81,\ 100,\ 140,\ 152,\ 210,\ \cdots;\ \cdots\\
53,\ 89,\ 151,\ 173,\ 251,\ \cdots;\ 46,\ 69,\ 111,\ 121,\ 161,\ \cdots;\\
\quad 70,\ 105,\ 154,\ 171,\ 236,\ \cdots;\ 135,\ 196,\ 276,\ 306,\ 376,\ \cdots;\ \cdots\\
\qquad \cdot \qquad \cdot \qquad \cdot \qquad\qquad\qquad \cdot \qquad \cdot \qquad \cdot
\end{array}$$

It is proposed to form permutations of the natural numbers

by interchanges among the elements of this array and for the present purpose it is enough to consider only exchanges of elements in the same column and permutations of terms in the same element. For the purpose of this paper is to examine the consequences of the following

ASSUMPTION.—*The permutations of an ω-series can be well ordered,*

and, denoting by Π the ordinal type of this series of permutations, it is clear that a Π-series may be obtained in either of the above ways, e. g., by permuting the terms in the first element or by permuting the elements in the first column. Indeed, if we select from the series of primes those whose numbers in the series are primes, the permutations of the whole set of natural numbers can be put into one-to-one correspondence with those obtained by only transposing pairs of consecutive members of the series

$$3, 5, 11, 17, 31, 41, 59, 67, 83, 109, \cdots$$

i. e., $1, 2, 5, 4, 3, 6, \cdots; 1, 2, 3, 4, 11, 6, 7, 8, 9, 10, 5, 12, 13, 14, 15, 16, 31, 18, 19, \cdots$, etc. Another way of obtaining a Π-series is by exchanging the first element of each column with the lower elements in the same column. It is easy to exhibit the one-to-one correspondence in this case by assigning to each nth digit (i, n) in a given permutation of the natural numbers the interchange of the first with the ith element in the nth column of our matrix. Thus to the permutation

$$2, 1, 4, 3, 6, 5, 8, 7, \cdots$$

will correspond that obtained by raising to the first place the second, fourth, third, . . . elements in the first, third, fourth, . . . columns respectively, that is, the permutation

$$1, 2, 7, 4, 13, 6, 3, 8, 9, 10, 23, 70, 5, 14, 15, 16, 29, 105, 19, 20, 21, 22, 11, 81, 25, 26, 154, 28, 17, 171, 43, 32, 33, 34, 35, 100, \cdots.$$

Obviously the same reasoning would hold if in each column we should exchange the series of first terms with the other series of nth terms. Moreover, both sets of transpositions may be performed simultaneously and yield a Π-series, since we get nothing but permutations of the natural numbers and clearly they are all different. In like manner, although we obtain a Π-series by permuting elements of a single column, we get no more by doing it in different columns independently. Similarly if we permute terms of a single element or make such permutations independently in different elements or if we combine this process with that of permuting elements in the

same column, in each case the resulting set of permutations may be written as a Π-series.

We will now show that our assumption leads to a set of permutations of the natural numbers which is equivalent to the whole set but can not be written in a Π-series. In any well-ordered set every element is an Nth term in some ω-series ($N = 1, 2, \cdots$). For a given N the set of all Nth terms shall be called the elements of the Nth *kind*. Permute independently the terms in different elements of our matrix in all possible ways and let $[P]$ denote the resulting set of permutations arranged in a Π-series.

There exists a permutation of the natural numbers which differs in every nth column of its matrix ($n = 1, 2, \cdots$) *from every element of the nth kind in* $[P]$. The elements of the nth kind in $[P]$, being by hypothesis part of a well-ordered set, form also a well-ordered set $[P^{(n)}]$ of ordinal type $\Pi^{(n)} \leqq \Pi$. The proposition will be proved by establishing the following

LEMMA. *For each value of n there exists a permutation of the natural numbers which differs in every Nth element in the nth column of its matrix from every element of the Nth kind in* $[P^{(n)}]$.

It is not necessary actually to produce such permutations; it is sufficient to show that they exist. This is easily done by observing that the elements of the first kind in $[P^{(n)}]$ do not use up the $\Pi^{(n)}$-series of its totality and hence do not exhaust the Π-series of permutations available for the first element of the matrix. Hence there is a permutation of the natural numbers differing in the permutation performed on its first element from every element of the first kind in $[P^{(n)}]$ and *at the same time* different for the same reason from every element of the Nth kind in $[P^{(n)}]$ in the permutation performed on the Nth element ($N = 2, 3, \cdots$) in the first column of its matrix. The same reasoning holds independently for every nth column ($n = 2, 3, \cdots$) and, of course, there are in each case an infinity of such permutations P' outside of $[P]$. So we have, on the basis of our assumption, two sets, $[P]$ and $[P] + [P']$, which are both in one-to-one correspondence with the whole set of permutations of the natural numbers but can *not* be put into one-to-one correspondence with each other.

Whence the assumption is false.

McPHERSON, KANSAS.

RELATIONS AMONG PARAMETERS ALONG THE RATIONAL CUBIC CURVE.

BY PROFESSOR J. E. ROWE.

(Read before the American Mathematical Society, April 24, 1915.)

Introduction.

THE purpose of this paper is to give the proofs of two new theorems concerning relations among sets of parameters along the rational plane cubic curve. The first theorem concerns a projective relation possessed by the parameters of the four residual points in which the osculant conic at any point meets the cubic. The second theorem defines the relation which exists among the parameters of the four tangents drawn from any point of the plane to the rational cubic. The proof depends upon a method of deriving the parametric equations of the node of the rational cubic.

We shall call the rational plane cubic the R^3, and write its parametric equations in the form

$$x_i = (a_it)^3 \equiv a_it^3 + 3b_it^2 + 3c_it + d_i \quad (i = 0, 1, 2). \tag{1}$$

§ 1. *The Osculant Conic and the Associated Theorem*

A point on the osculant conic of the R^3 at a point whose parameter is t' is defined by the equations

$$x_i = (\alpha_it')(\alpha_it)^2 \equiv (a_it' + b_i)t^2 + 2(b_it' + c_i)t + (c_it' + d_i) \quad (i = 0, 1, 2). \tag{2}$$

The equation* of the osculant conic at t has the form

$$[4|abx||bcx| - |acx|^2]t^4 + [4|abx||bdx| - 2|acx||adx| + 2|acx||bcx|]t^3 + (\cdots)t^2 + (\cdots)t + (\cdots) = 0. \tag{3}$$

In particular, if $t = 0$, (3) becomes

$$4|bcx||cdx| - |bdx|^2 = 0. \tag{4}$$

If the values of x_i from (1) are substituted in (4), and the factor t^2 is removed, the result is

$$[4|abc|\,|acd| - |abd|^2]t^4 + 12|abc|\,|bcd|t^3 + 6|abd|\,|bcd|t^2 + 4|acd|\,|bcd|t + 3|bcd|^2 = 0. \tag{5}$$

* J. E. Rowe, *Messenger of Math.*, No. 512 (Dec., 1913), pp. 118–119.

The roots of (5) are the parameters of the residual points in which the conic (4) cuts the R^3. The invariant S of (5) is

$$(6)\quad 12|abc|\,|acd|\,|bcd|^2 - 3|abd|^2|bcd|^2 - 12|abc|\,|acd|\,|bcd|^2 + 3|abd|^2|bcd|^2,$$

which vanishes identically. No restriction was imposed by selecting the osculant conic of the R^3 at the point whose parameter is 0; i. e., if $S = 0$ for (5) derived from this particular osculant conic, the same projective relation $S = 0$ will exist for every quartic found by selecting a particular value for t in (3). This result may be stated in the form of

THEOREM I. *The parameters of the four residual points of intersection of the R^3 and any osculant conic form a self-apolar set.*

§ 2. *The Equation of the Node of the R^3 and its Significance.*

Every line section of the R^3 has three parameters which are apolar to the three flex parameters. The two nodal parameters and *any* other are apolar to the three flex parameters, for any line on the node determines the two nodal parameters and one other; this third parameter may be used to distinguish one line on the node from another.

From any point in the plane four tangents can be drawn to the R^3. The four parameters of the tangents from the point x_i to the R^3 are the roots of the quartic*

$$(7)\quad (\alpha t)^4 \equiv |abx|t^4 + 2|acx|t^3 + [adx + 3|bcx|]t^2 + 2|bdx|t + |cdx| = 0.$$

The three flex parameters† are the roots of

$$(8)\qquad (\beta t)^3 \equiv |abc|t^3 + |abd|t^2 + |acd|t + |bcd| = 0.$$

The two nodal parameters may be found as the pair which, together with an arbitrary third, are apolar to (8); this amounts to finding the Hessian of (8). Hence, they are given by

$$H \equiv \begin{vmatrix} 3|abc| & |abd| & |acd| \\ |abd| & |acd| & 3|bcd| \\ 1 & -t & t^2 \end{vmatrix} = 0.$$

---

* Loc. cit., p. 119.

† For equation (8) and its Hessian H see Salmon's Higher Plane Curves, third edition, pp. 187–188.

The polar of (8) with respect to (7) is, in symbols,

$$(\beta\alpha)^3(\alpha t) = 0, \tag{9}$$

which expanded becomes

$$\begin{aligned} (10)\quad [3|abc|\,|bdx| - |abd|(|adx| + 3|bcx|) + 3|acd|\,|acx| \\ - 6|bcd|\,|abx|]t + [6|abc|\,|cdx| - 3|abd|\,|bdx| \\ + |acd|(|adx| + 3|bcx|) - 3|bcd|\,|acx|] = 0. \end{aligned}$$

For a given value of t, (10) is the equation of a line; the intersections of this line and the R^3 may be found by substituting from (1) in (10). If $t = t_1$ in (10) and the values of x_i from (1) are substituted in the equation of the line so derived, the result may be thrown into the form

$$H(t - t_1) = 0. \tag{11}$$

This is sufficient to show that (10) is a pencil* of lines on the node, and may be said to envelope the node. The parametric equations of the node as a rational curve of the first class are found by identifying (10) and the equation

$$(\xi x) = \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 = 0;$$

these assume the form

$$\begin{aligned} (12)\quad \xi_i = [3|abc|(b_\mu d_\nu) \cdots - 6|bcd|(a_\mu b_\nu)]t \\ + [6|abc|(c_\mu d_\nu) \cdots - 3|bcd|(a_\mu c_\nu)] \quad (i,\ \mu,\ \nu = 0,\ 1,\ 2). \end{aligned}$$

Also (10) as it stands may be considered the equation of the node. This shows how the parameters of the four tangents from any point of the plane are related. The equation (10) for a particular set of the x_i yields a value of t which together with the three flex parameters constitutes a set of four apolar to the quartic found by substituting the set of the x_i in (7). The result may be stated as a theorem.

THEOREM II. *The parameters of the four tangents from any point P of the plane to the R^3 are apolar to the set of four composed of the three flex parameters and the parameter of the other point cut out of the R^3 by the line joining P to the node of the R^3.*

PENNSYLVANIA STATE COLLEGE,
March, 1915.

---

* Compare with W. Gross, *Mathematische Annalen*, vol. 32 (1888), pp. 144–145.

VALLÉE POUSSIN'S COURS D'ANALYSE.

Cours d'Analyse Infinitésimale. Par CH.-J. DE LA VALLÉE POUSSIN. Tome 1, troisième édition considérablement remaniée, et tome 2 remaniée. Louvain, Dieudonné, 1914. 9 + 452 pp. and 9 + 464 pp.

IN the two four hundred and fifty page volumes of this Cours the author has in mind two classes of readers. There are, first, those who desire to acquire an accurate working knowledge of the calculus stripped as far as possible of those subtilities which are repellant and useless to the engineer and physicist. This part of the book is printed in large type and follows in the choice of topics the general outline of the traditional French Cours, except that the space devoted to the treatment of Fourier's series is somewhat greater and convergence proofs are given. The handling throughout is clear, elegant, and concise; the various topics are illustrated by numerous carefully chosen examples selected with rare pedagogic skill to develop a real understanding of the text.

The rest of the Cours, printed in smaller type, is addressed to a different class of readers, those who wish to get at the fundamental principles of modern analysis. These last editions show that both volumes have undergone considerable alterations and improvements, proofs have been recast and expanded and the books, though excellent in the first edition, have been greatly improved.

§§ 8–10 deal with sets in general, and it would be hard to find anywhere so lucid and compact a presentation of the fundamental ideas involved. § 10 is concerned with the Borel-Lebesgue theory of measure and establishes the important results of Borel and Lebesgue, the methods of proof being essentially those later used by Vitali in his paper "Sui gruppi di punti" in volume 28 of the *Rendiconti del Circolo Matematico di Palermo.*

§ 12 deals with measurable functions and it is shown that practically all convergent processes applied to measurable functions lead to measurable functions.

§ 13 is concerned with functions of limited variation, destined later to play such an important rôle in the theory of Lebesgue integrals, and ends with a section on Vitali's *absolutely* con-

tinuous functions, the fundamental importance of which in Lebesgue's theory does not seem to be generally recognized.

The above constitutes an introduction to the subsequent treatment, which now begins with the elementary theory of derivatives followed in § 111 by arithmetic demonstrations of the familiar properties (due largely to Dini) of the four *derivates*. Here the book contains certain new matter, the most striking being the author's generalization of a famous theorem of Scheeffer concerning the determination of a continuous function when one of its derivates is known except for a null set.

The statement of this theorem is modeled somewhat after Scheeffer's, which has been criticized by Schoenflies as being illogical. We give it in the author's own words.

Si, dans un intervalle (a, b), *deux fonctions* $f_1(x)$ *et* $f_2(x)$ *ont leurs nombres dérivés supérieurs à droite*: 1° *finis en chaque point sauf peut-être dans un ensemble* E_1, *et* 2° *égaux sauf peut-être dans un ensemble de mesure nulle, les deux fonctions ne different que par une constante à moins que* E_1 *ne contienne un ensemble parfait.*

As stated, the theorem might seem self-contradictory for if $f_1 \equiv f_2 + c$ then the derivates will be *everywhere* equal.

This generalization of Scheeffer's theorem is not quite as general as it seems, for W. H. Young has shown that the set of points E_1 is either denumerable or has the power of the continuum, so that the theorem only holds when E_1 is denumerable. The generalized theorem admits, as the author points out, a sort of inverse, though in the proof given on page 102 the functions $y = \psi(x)$ and $x = \psi^{-1}(y)$ are not both continuous, as stated; one of them is not even singly valued.

The theory of Riemann integrability receives an elegant but very summary treatment and the author begins his exposition with the remark, "cette théorie n'a plus guère qu'une importance historique, car elle rentre comme cas particulier dans celle de Lebesgue, qui sera étudiée dans le chapitre suivant." A statement true only of proper Riemann integrals.

Chapter seven begins the systematic treatment of Lebesgue's integrals and under the general theory gives the relation to Riemann's integrals and six sufficient conditions for the validity of the equation

$$\int_a^b \lim_{n=\infty} f_n(x)dx = \lim_{n=\infty} \int_a^b f_n(x)dx,$$

wherein the greater simplicity of the conditions for Lebesgue integrals over those for Riemann's is amply evidenced. Summable functions are considered from the start and it is shown that, unlike the Riemann improper integral, the Lebesgue integral can always be defined as the limit of a sum $\Sigma l_i e_i$, and that the integral of a summable function has a derivative equal to the integrand except over a null set (presque partout—which we may translate "almost everywhere"). The treatment from now on shows a marked departure from that of Lebesgue in that it is less elementary but easier reading.

The author has devised a method of *majorating* ϕ_1 and *minorating* ϕ_2 functions such that

$$\phi_1 > \int_a^x f(x)dx > \phi_2$$

and

$$\Lambda\phi_1 > f(x) > \Lambda\phi_2$$

and with their aid establishes the capital theorem that the Λ of any monotone function f is summable and that its integral differs from $f(x)$ by a function $V(x)$ defined as the *variation* of $f(x)$ over the set of points E where $\Lambda f(x)$ is infinite.

Finally he establishes the corner stone of the theory by showing that the *necessary* and *sufficient* condition that

$$f(x) - f(a) \equiv \int_a^x \Lambda f(x)dx$$

is that $f(x)$ be *absolutely* continuous, as pointed out by Vitali in his paper in the *Atti della R. Accademia delle Scienze de Torino*, 1905, "Sulle funzioni integrali."*

Whether or not all functions with summable derivates belong to the class of functions of limited variation is left open, though it seems that this could have been answered in the affirmative from the theorems demonstrated in the text.

Original matter is taken up in § 267, where integration by substitution is considered. Here the results are of remarkable simplicity and generality, the final result being: If $f(x)$ is a

* The question of priority here is doubtful. Schoenflies, in the second volume of his Bericht, refers to papers by Levi of about the same date and does not mention Vitali, of whose papers he does not seem to be aware. A theorem of the author's in the first edition *practically* amounts to the condition of Vitali.

limited* summable function in (a, b) and $x = \phi(t)$ an absolutely continuous function of t such that x varies from $x_0 = \phi(t_0)$ to $X = \phi(T)$ always remaining in ab, then

$$\int_{x_0}^{X} f(x)dx = \int_{t_0}^{T} f(\phi)\phi'(t)dt,$$

where $\phi'(t)$ is defined to be zero in the null set where it does not exist.

The proof here could have been somewhat simplified if the author had made use of the absolutely continuous function $\phi(x)$ used at the top of page 101.

The chapter closes with an investigation of the properties of the second generalized derivates and derivatives. The proofs make liberal use of geometric intuition, though their arithmetization would probably not be difficult. Condition (K), § 274, is described in too summary a manner and it is not at once evident how it differs from merely postulating a right-handed derivative for the function $F(x)$. Finally it is shown that if $f(x)$ is summable and between (or equal) to the upper and lower right (left) generalized second derivates of $F(x)$,

$$F(x) \equiv \int_{a}^{x} dx \int_{0}^{x} f(x)dx + Lx + n.$$

These theorems play an important rôle in the theory of Fourier's series and the author has shown their power in his own researches to which reference will be made later.

In §§ 342 et seq. continuous and closed curves are treated and the author fills in the lacunæ of his proof that a closed continuous curve without double points divides the plane into two parts. Here certain topological theorems concerning *chains* play a leading rôle. A *link* is a connected region of the plane bounded by an uncrossed outer polygon and containing various *holes* bounded by polygons of the same sort; a *regular open chain* consists of a series of links such that *consecutive* and only consecutive links have points in common. The theorem to be established is that a closed continuous curve determines a sequence of thinner and thinner closed regular chains containing the curve in the links. The only

* The text has it *finite*, but the lemma on which the proof rests is not true unless the function is limited, as has been pointed out by Dr. Dunham Jackson.

postulate (not explicitly stated) needed to carry out the proof is that a closed uncrossed polygon of N sides divides the plane into two parts.

Rectifiable and quadrable curves are then taken up and the necessary and sufficient conditions are obtained together with the formula

$$s = \int_{t_1}^{t} \sqrt{x'^2 + y'^2}\,dt,$$

where the integral of Lebesgue is used and $x(t)$ and $y(t)$ are absolutely continuous functions of t, and the formulas for area

$$\int_{t_1}^{t} xy'dt, \quad - \int_{t_1}^{t} yx'dt, \quad \tfrac{1}{2}\int_{t_1}^{t} (xy' - yx')dt,$$

where the integrals are Lebesgue's and the only hypothesis is that the function (functions) whose derivatives figure shall be absolutely continuous.

This volume closes with three sections on *quasi-uniform* convergence—a name proposed by Borel to take the place of Arzelà's *convergenza uniforme a tratti*—and an elegant proof of Arzelà's celebrated theorem that *the necessary and sufficient condition that the limit of a convergent sequence of continuous functions be continuous is that the convergence be quasi-uniform.* Arzelà's necessary and sufficient condition for the termwise integrability of a series using Riemann integrals is not touched upon because the matter is so much simpler when Lebesgue integrals are used. The sufficient conditions in the latter theory are stated and proved.

A note supplementary to the second edition of volume two has been added dealing with the uniqueness of trigonometric developments,* where among others the following interesting theorem is proved:

If the coefficients of a trigonometric series approach zero with $1/n$ *and the upper and lower limits of* $S_n(x)$ *for* n *infinite are summable and finite save in a null set* E, *the trigonometric series will be a Fourier series if* E *is not of the power of the continuum.*

Here, as in the case of Scheeffer's theorem, the statement holds only in the case that E is a denumerable set.

In connection with this theorem it is of interest to note that

---

* Taken from two papers by the author in the *Bulletin de l'Académie royale de Belgique*, No. 11 (1912), No. 1 (1913).

Hugo Steinhaus has constructed a trigonometric series *everywhere* divergent where coefficients approach zero as limit. If this is a Fourier series, it is an example of the long sought everywhere divergent Fourier series.

The second volume is largely devoted to functions of several variables and after taking up double integrals from the more elementary standpoint proceeds to establish for the more advanced reader the leading theorems in the Riemann theory in the author's usual terse and elegant fashion, followed immediately by an extensive exposition of Lebesgue multiple integrals, where the theory follows, in a way, the broad outlines of functions of a single variable but where new concepts must be introduced, such as *density of a set in a point* and a generalized definition of derivatives. A theorem of Vitali's somewhat resembling the Heine-Borel theorem is then established and the important theorem:

An additive absolutely continuous function of a set has almost everywhere a finite and determinate derivative and is the indefinite integral of this derivative.

This is the analogue of the theorem already stated for absolutely continuous functions of a single variable.

The theorems of Lebesgue and Fubini on iterated double integrals follow and illustrate in a striking manner the greater simplicity and generality of the sufficient conditions in Lebesgue's theory over those in Riemann's.

The chapter closes with a generalization of Green's theorem where it is shown that

$$\int_C Pdx + Qdy = \int\int_D (Q_x' - P_y')dxdy$$

provided

1°. that P and Q are continuous inside of C (over D).

2°. that P is absolutely continuous in y and Q absolutely continuous in x.

3°. P_y' and Q_x' are summable in D.

The author goes on to remark that the absolute continuity of P and Q would be secured if $\Lambda_y P$ and $\Lambda_x Q$ are finite. The *limitedness* of these derivatives is a sufficient condition for both 2° and 3°. Applied to the standard proof of Riemann of Cauchy's theorem that

$$\int_C f(z)dz = 0, \quad f(z) \equiv u(xy) + iv(xy)$$

when $f(z)$ is analytic inside C and on the boundary, this theorem shows that we need not even assume the *existence* of $f'(z)$ in D, but need only assume that u and v have *bornées* first derivatives satisfying the Cauchy-Riemann partial differential equations (cf. Goursat's well-known proof).

Next follows a beautiful chapter on the approximate representation of analytic functions, which closes with the treatment of Fourier's series in which the most important results of Dirichlet, Riemann, Dini, Cantor, Fejér, and Lebesgue are established.

In the compass of such a review, it is impossible to point out all the merits of these volumes, so rich in varied topics, so lucid in exposition and elegant in presentation. A unique feature of the book is that it does for Lebesgue's integrals what Jordan did for Riemann's theory.

Aside from his lectures delivered at the Collège de France under the Peccot foundation and published in 1904 in the Borel Series of Monographs on topics in the theory of functions, and his Lectures on Trigonometric Series, Lebesgue has published no systematic exposition of his ideas, contenting himself with the publication of numerous papers in various journals and transactions. The great value of the theories with which he has enriched analysis makes a systematic presentation of them a matter of great importance and we owe Professor Vallée Poussin a profound debt of gratitude not only for having completed this theory in many essential particulars but for his masterly presentation of it as a whole. With such a treatise available, these theories will become the common property of all mathematicians and, while certain simplifications and improvements in the demonstrations will come about in time, the outline and main structure has been definitely fixed. From the simple but genial idea that a generalization of the integral concept might come from dividing up the interval of variation of the dependent variable (instead of the independent variable's field as in Riemann's theory) the genius of Lebesgue has created a large and growing domain of analysis whose great importance cannot as yet be accurately estimated, but whose value in dealing with the

more recondite problems of analysis is amply exemplified in these two volumes.

M. B. PORTER.

AUSTIN, TEXAS.

EXTRACT FROM A LETTER FROM PROFESSOR DE LA VALLÉE POUSSIN.

Cher monsieur et collègue,

Vous citez une objection de Schoenflies à propos de l'énoncé du théorème: Si dans un intervalle (a, b) deux fonctions $f(x)$ et $f_1(x)$ ont leurs nombres dérivés supérieurs à droite: 1° finis en chaque point sauf *peut-être* dans un ensemble E_1 et 2° égaux sauf *peut-être* dans un ensemble de mesure nulle, les deux fonctions ne diffèrent que par une constante à moins que E_1 ne contienne un ensemble parfait.

M. Schoenflies trouve cet énoncé contradictoire parce que deux fonctions qui ne diffèrent que par une constante ont la meme derivée *partout.*

Le mot *peut-être* que j'ai souligné dans l'énoncé a manifestement un sens subjectif et je refère à l'incertitude où nous pouvons être sur l'égalité ou la non égalité des nombres dérivés. L'objet du théorème est d'ailleurs précisément de lever cette incertitude. Voilà du moins ce que j'ai pensé.

L'objection de M. Schoenflies est d'ailleurs contestable en elle-même. Il n'est pas faux de dire que deux fonctions qui ne diffèrent que par une constante ont des derivées égales *sauf dans un ensemble de mesure nulle.* Car les derivées peuvent être infinies dans un ensemble de mesure nulle et on est en droit de dire que deux quantités infinies ne doivent pas être considérées comme égales. Je ne tiens d'ailleurs à cet argument que contre M. Schoenflies.

Vous faites remarquer encore que la généralisation du théorème de Scheeffer est moins grande qu'elle ne paraît parce que M. Young a démontré que E_1 est ou bien dénombrable ou bien a la puissance du continu. Donc, ajoutez-vous, le théorème vaut seulement si E_1 est dénombrable.

Pour que cette conclusion fût exacte, il faudrait démontrer que E_1 est ou bien dénombrable ou bien contient un ensemble parfait. Je crois bien que cela est vrai de tout ensemble mesurable (B) mais est-ce la meme chose que le théorème énoncé de M. Young?

Je vous signale enfin que l'ouvrage sur lequel vous faites

rapport a été entièrement brûlé le 27 août dernier à Louvain, avec tout le magasin de mon éditeur, le 3e jour de l'incendie de cette ville par l'armée allemande.

Je vous prie, cher collègue, d'agréer l'expression de mes remerciments et l'assurance de mon entier dévouement,

C. DE LA VALLÉE POUSSIN.

CAMBRIDGE, MASS.,
May 11, 1915.

The objection to the formulation of Scheeffer's theorem referred to in Schoenflies' Bericht, volume 2, page 317, was directed at Scheeffer's statement of it. The reviewer was under the impression that even as stated above the *peut-être* might be objectively interpreted.

THE REVIEWER.

ENUMERATIVE GEOMETRY.

Lehrbuch der abzählenden Methoden der Geometrie. By H. G. ZEUTHEN. Leipzig, Teubner, 1914. xii + 394 pp. Price (cloth) 17 Marks.

IN the preface to his Lehrbuch Zeuthen expresses his gratitude to the publishers, "that the researches, which I have delighted to pursue from youth to an advanced age, may now appear in their full sequence." The mathematical world also has reason for hearty gratitude, not only to the Teubner firm, of whose family of publications this book is a very worthy member, but much more to Zeuthen himself, that he has produced a book summing up most carefully and elegantly both the chief results and the most fertile methods of enumerative geometry.

Zeuthen must have wished more than once in writing this work that a book were not essentially a one-dimensional configuration. The greater part of the book could have been displayed most satisfactorily in a plane with an axis of methods perpendicular to an axis of subjects. This arrangement being impossible, the author chose to make his work primarily a text on methods, and so to devote each chapter to a single method or group of methods. Within each chapter the results are grouped according to the configurations to which they apply, usually in the following order: plane curves, surfaces (in S_3), space curves, line configurations. The defect

of one-dimensionality is then remedied by a short table of contents at the end, where all sections on the same figure are grouped together, and by a very complete set of cross-references in the text. We have, then, a treatise which may be regarded either as an exposition of the enumerative methods of projective geometry or as a very extensive account of the results obtained by those methods in the case of the most important types of figures in three-dimensional space.

The book is essentially one on projective geometry. To be sure, some of the apparatus of the geometry of birational transformations is developed. The genus of a curve, the arithmetic genus and the Zeuthen-Segre invariant of a surface are defined; but both definitions and applications are in terms of projective characteristics of the figures. On the other hand, the author descends to affine and metric geometry with readiness; a quite unexceptionable procedure, especially since he draws attention to the fact that these are but special cases of projective geometry.

The enumerative methods described by Zeuthen are applied almost exclusively to figures defined by algebraic equations. In a few places, to be sure, it is remarked that a method is applicable to systems of curves defined by means of certain algebraic differential equations—as (Article 163) in determining the number of curves of a system having contact of given order with a given curve; but even there the problem to be solved originally is purely algebraic.

A problem of enumerative geometry is one which asks the number of points, lines, curves, etc., of a system which fulfil certain conditions. Thus the number of intersections of two plane curves of given order can be regarded either as the number of the ∞^2 points in the plane which lie on both curves, or as the number of the ∞^1 points of one curve which lie on the other. Every such problem can be defined also as a determination of the number of solutions, that is, of the degree, of an algebraic equation.

The fundamental principle of enumerative geometry is the law of the "preservation of the number" (Erhaltung der Anzahl). That was stated by H. Schubert* in the following form: Let there be a variety of ∞^n objects on Γ, which shall be imposed a condition of dimension n, defined by assigned relations between Γ and another variety Γ'. Then the (finite)

* H. Schubert, Kalkül der abzählenden Geometrie (1879).

number of objects Γ satisfying that condition remains unaltered, however we may particularize Γ', provided the number remains finite. There arise a very considerable number of difficulties and dangers in the use of the methods based on this principle, and the care with which Zeuthen has treated them is worthy of all praise.

At the outset he lays down three necessary precautionary rules, which are never lost sight of. The first states that every case of a general problem must be considered as a limit of a continuous series of cases. An obvious corollary of this is expressed later in these words (Article 158): "In the application of formulas which express the number of configurations fulfilling diverse conditions, one must retain for each condition the exact meaning which was laid down at its introduction into the formulas." Thus, an arbitrary line through a double point of a plane curve is to be considered as a tangent to the curve if a tangent is defined as a line which has two coincident intersections with it; this is confirmed by the fact that such a line is the limit of a tangent to a curve which "acquires" a double point. On the other hand, it is not to be taken as a tangent if, the curve being defined by an equation in line coordinates, a tangent is regarded as an element of it. The second rule is that, when coincident solutions of a problem are counted, care must be taken that such solutions are counted the due number of times. Thus, the line from an arbitrary point of the plane to any double point of a plane curve—if we take the first definition of a tangent—counts as two tangents; this again would seem natural from the aspect of a curve about to acquire a double point. The third rule (this one expressly included in Schubert's enunciation) is that an enumerative formula loses validity and significance if the objects which it normally enumerates turn out, in a particular case, to be infinite in number. Thus, the unique value of dy/dx usually proves the existence of a unique tangent at a point of a curve; at a singular point dy/dx assumes the form $0/0$, and there is a tangent through the point in every direction—if we define a tangent rightly.

Study and Kohn* remarked in 1903 that the principle of the preservation of the number, as stated by Schubert, is not

* Study, Geometrie der Dynamen, p. 378. Kohn, "Ueber das Princip von der Erhaltung der Anzahl," *Archiv der Mathematik und Physik* (3), Bd. 4, pp. 312–316.

always valid; Kohn, indeed, said that it was "in a certain sense incurably ill." That is, a condition expressed in general form may be fulfilled in a certain finite number of cases; whereas, for certain particularizations of these conditions, the number of solutions, though remaining finite, may be increased. The source of this phenomenon is the possibility that the conditions may be fulfilled in different ways. The particular problem examined by Study and, in his article on the principle of the preservation of the number, by Severi,* is that which asks the number of projectivities of a line which transform into itself a given group of four points. If the cross-ratio of the group is not a cube root of -1, the number of projectivities is 4. If the cross-ratio is -1, there are 8 solutions; and if it is an imaginary cube root of that number, there are 12. The excess comes, as Severi remarks, from the possibility, which exists for harmonic and equianharmonic groups alone, that a projectivity may leave unchanged some of the points and interchange the others.

Severi stated and proved a theorem giving the conditions under which Schubert's principle can be safely applied. His briefer enunciation of it is this: "The principle of Schubert holds only for those conditions which can be resolved into sums of irreducible conditions of the same dimension." Thus, in the example which Severi cites from Study, the condition that a group of four points of a line be left invariant by a projectivity is the sum of four conditions: (α) that a projectivity be involutory and interchange all of a group of four points, (β) that it be involutory and permute two points of a group, not moving the others, (γ) that it be cyclic of order 3 and permute a group, (δ) that it be cyclic of order 4 and permute a group. (β), (γ), (δ) are of higher dimension than (α). Thus Severi amputated the incurable member, and left us the certainty that the body, after the operation, was quite free from disease. It was a beautiful and valuable piece of work.

And yet Zeuthen, while he says expressly (Article 189) that he has read Severi's article, does not quote his theorem. Severi's restriction of the availability of Schubert's principle achieves no more than the precautions which Zeuthen teaches. Consider, in particular, the problem examined by Severi.

* Severi, "Sul principio della conservazione del numero," *Rendiconti del Circolo Matematico di Palermo*, vol. 33 (1912), pp. 313–327.

The projectivities of the line which transform harmonic and equianharmonic groups into themselves, while interchanging only some of their points, can not be the limits of projectivities which transform into themselves groups with cross-ratios approaching a cube root of -1; for the latter transformations interchange either all or none of the points of the groups. The projectivities applicable to groups of special cross-ratios only are, then, by Zeuthen's first rule, expressly excluded from those enumerated. It seems possible, then, to do without Severi's theorem; yet that theorem is so elegant, clear, and simple, that it will prove a great aid to workers in enumerative geometry, and might well have been for that reason quoted in the present work.

An interesting and practically valuable use of reasoning from a particular to a more general case is offered by Zeuthen's justification of deductions made from real figures. He reminds us (Article 47) that all possible cases of a theorem depend on the values of a certain number of parameters, and that a real figure represents any one of an infinite number of sets of values of those parameters—all within certain limits. Enumerative properties observed in the figure, holding as they do for an infinite number of values of the parameters, will hold for all values—even such as make the figure imaginary. In this manner he obtains the fact that if the points of tangency of a single branch of a curve with a double tangent approach each other, two points of inflection also approach each other, and all four coincide where the curve acquires four-point contact with the line. That some caution is needed in reasoning from figures is shown by consideration of the number of inflections absorbed in a double point. The figure shows that a curve about to have a double point has at least two inflections which approach that point; analysis proves, however, that there are in truth six, four of which must be imaginary.

So much for examples of the care which Zeuthen exercises and inculcates,—surely the most important quality in a treatise on his subject. Praise should also be bestowed on the almost universal clearness of exposition. If exceptions are noted, let it be remembered that exceptions are far to seek. The first sentence of Article 32 reads thus: "Da die Komplexe einer gegebenen Ordnung m eine zusammenhängende Menge bilden, kann man einen solchen in abzählenden Untersuch-

ungen spezialisieren, z. B. in der Weise, dass man verlangt, die Strahlen sollen m gegebene Gerade schneiden." On its face, this demands that each ray cut all of m given lines; while the meaning is, of course, that it cut some one of them. Another inaccurate passage occurs in the third paragraph of Article 172. The system there spoken of should be determined by one line and three points.

The references to sources are very few. For the most part, Zeuthen contents himself with mention of the bibliography in his article in the German Encyclopedia (III C 3) on the same subject. It is unfortunate that for the many developments of line geometry in the present treatise the Encyclopedia article has no mention.

Zeuthen's modesty gives another cause for regret. It would serve the reader better to become familiar with "Zeuthen's formula" than with the "allgemeiner Geschlechtsatz"; with the "Zeuthen-Segre invariant" than with "I."

A laudable feature of the book is the great number of exercises. They cover applications of all the principles developed and vary widely in difficulty—some being, as the author says, suitable even for doctor theses.

Chapter I is introductory. Its first half discusses the methods and aim of enumerative geometry, and gives definitions. The second half treats the method of determining the number of solutions falling together. An ingenious scheme, due to Zeuthen, for fixing this number appears in various forms in different sections of the book. Its first form is as follows: "The number of intersections of a line a with a curve at a point A can be defined as the sum of the orders of infinitesimal segments between a and the intersections of the curve with l, a straight line making a finite angle with a, at a distance from A that is infinitesimal of the first order." There follow a clever proof of Bézout's theorem and one of Halphen's theorem concerning the point multiplicity and line multiplicity of an element of a curve.

The first part of Chapter II deals with direct applications of the principle of preservation of the number, such as the determination of the enumerative properties of polar curves, of Hessians, of Reye's complex. By the use of Schubert's principle, theorems are deduced concerning general curves and surfaces from the consideration, as special cases, of degenerate ones. The degenerate plane curve of degree n

may be a set of n straight lines. It may be the projection of the curve on a line in its plane; that is, the line counted n times, with a certain number of vertices (Scheitel). In either case the simplification is so great as to create a highly efficient engine.

The next section treats problems with an excessive and hence infinite number of solutions. The problems solved concern largely the number of points necessary for determining completely various configurations,—plane curves, surfaces, curves on surfaces. Two paragraphs are devoted to Poncelet's "closing theorems" (Schliessungssätze), which deal with the question of whether polygons whose sides and vertices are respectively tangents and points of certain conics belonging to a single pencil are closed or open. Problems on the closing of polygons whose sides and vertices fulfil various conditions have been of unusual interest to Zeuthen; he treats them, indeed, in eight sections of the book.

The section entitled "Problems with no solutions" is largely devoted to applications of the principle that an algebraic function which is not constant must be able to assume all values. This principle is remarkably fertile; it furnishes, for instance, proofs of the constancy of various cross-ratios, among them that of lines from a variable point of a conic to four fixed points on it, and that of the tangents to a plane cubic from a variable point on it.

Chapter III is concerned with applications of Zeuthen's formula (allgemeiner Geschlechtsatz). If between the points of two curves c_1, c_2 of genus p_1 and p_2 respectively there exists an (α_1, α_2) correspondence; if, further, the number of cases in which two of the α_1 points corresponding to a point of c_2 coincide is η_1, and the inverse number is η_2, the formula is

$$\eta_2 - \eta_1 = 2\alpha_1(p_2 - 1) - 2\alpha_2(p_1 - 1).$$

An immediate consequence is Riemann's theorem of the equality of the genera of two curves in (1, 1) correspondence with one another. There is a careful discussion of Plücker's equations and of the analysis of more complicated singularities of plane curves. Plücker's equations, together with Zeuthen's formula, offer a means for investigating the order, class, and singularities of a curve in correspondence with a given curve (for example, its evolute). There follows a first consideration of systems of curves. The author treats the application of

his and Plücker's formulas to space curves by means of their projections on a plane, and to surfaces by means of their circumscribed cones. A short section is given to the "Geschlechtsätze" for surfaces, and their application to curves on surfaces in (1, 1) correspondence. In the formulas in question the arithmetic genus and the Zeuthen-Segre invariant play the part taken by the genus of a plane curve in Zeuthen's formula.

The Cayley-Brill correspondence principle for points on a curve and analogous ones for points in a plane, on a surface, and in space, together with a wealth of applications of these theorems, form the subject matter of Chapter IV. In developing the Cayley-Brill theorem, Zeuthen defines the valence (Wertigkeit) k of an (α_1, α_2) correspondence between points of a curve of genus $p > 0$ by means of the formula

$$\gamma = \alpha_1 + \alpha_2 + 2kp,$$

γ being the number of self-corresponding points. He then proves that if the point P_1, taken k times, and the α_2 corresponding points P_2, each taken once, form the complete intersection of the ground-curve with a curve depending on P_1, then the value of k coincides with that obtained from the above equation. This order of development seems a little unnatural, but it has the great advantage of giving to negative and fractional valences equal rights with their more normal brothers. The final sections of this chapter are devoted to the correspondence principle for points on a surface, which Zeuthen announced in 1906. The form of this theorem is similar to that of Cayley-Brill, though naturally more complicated; the Zeuthen-Segre invariant takes the place, in a way, of the genus, and the valence has an exactly analogous interpretation. If one is to judge from the fruitfulness of its prototype, it should play an important part in the theory of surfaces.

The title of Chapter V is "Systems of Configurations." Systems of curves (in particular, of conics), of surfaces, and of correlations are treated; usually for the purpose of finding the number of elements of a system fulfilling given conditions. In the extended discussion of systems of conics, due attention and care are given to what one rather hesitates to speak of as Halphen's degeneracy. That is a conic which, in line coordinates, is a point, in point coordinates a line through the

point; and which, considered as a limit of conics of a system, baffles description by both point and line coordinates.

The closing chapter deals with the powerful and elegant methods of Schubert's symbolic calculus. We could wish for a more complete statement of the meaning of the symbolism; yet the chapter is well written, and presents perhaps the most interesting method treated in the book. The work ends with a section which opens the way to the application of Schubert's methods to four-dimensional geometry.

The typography of the book is good, but not quite up to Teubner's highest standard—the standard, for instance, attained in Zeuthen's Encyclopedia article. I give a partial list of misprints discovered, with genuine regret at ending thus a review of a treatise so important for both the science and the art of mathematics.

Page 74	line 29, 30	*for* (ab)	*read* (ac)
79	1	*for* c_1	*read* c_2
102	5	*for* c_2	*read* c_n
112	4	*for* $\frac{1}{2}(n-1)(n-2)=d-e$	*read* $\frac{1}{2}(n-1)(n-2)-d-e$
125	12	*for* e	*read* c
139	28	*for* Fläche	*read* Kurve
145		*for formula* (1)	*read* (3)
146		*for formulæ* (2), (2′)	*read* (4), (4′)
163	10	*for* m_1'''	*read* m_4'''
208	37	*for* $n(k-1)$	*read* $k(n-1)$
250	9	*for* P_4	*read* P_7
276	35	*for* m'	*read* m''
286	1	*for* τ_0	*read* η_0
286	6	*for* γ_n	*read* γ_{0-n}
298	last	*for* c_1	*read* c_3
301	30	*for* γ_1	*read* p_1
315	14	*for* $\alpha\mu+\alpha\mu'$	*read* $\alpha\mu+\alpha'\mu'$
317	27	*for* (4)	*read* (2)
329	31	*for* einen Punkt und drei Gerade	*read* eine Gerade und drei Punkte
329	33	*for* auf einer der gebenen Geraden	*read* auf der gegebenen Geraden
334	27	*for* [175]	*read* [174]
336	24	*for* $(\mu\mu'^2)+4\beta$	*read* $(\mu\mu'^2)=4\beta$
384	23	*for* $\zeta g_p=\zeta_p+\zeta G$	*read* $\zeta g_p=\zeta p+\zeta G$

BROWN UNIVERSITY.

EDWARD S. ALLEN.

MATHEMATICS IN AUSTRALIA.

The Teaching of Mathematics in Australia. Report presented to the International Commission on the Teaching of Mathematics. By H. S. Carslaw. Sydney, Angus and Robertson, 1914. (Agents: Oxford University Press, London and New York.) 8vo. 79 pp.

During the deliberations of the International Congress of Mathematicians at Rome in 1908, the necessary steps were taken to organize an International Commission on the Teaching of Mathematics, the members of which were to prepare or procure reports on the teaching of mathematics in different countries. Many of these reports were ready for the Cambridge Congress in 1912, but since then several more have appeared. At this writing, 18 countries have published 172 reports with a grand total of 11,186 pages. Germany has already issued 46 reports with nearly 4,000 pages; the tale is told in about a fifth of this space, each, by Austria with 13 reports, Great Britain with 34 reports, Switzerland with 9 reports, and Japan with 2 volumes. The reports of France and the United States each cover some 700 pages. Of more modest dimensions are, in order of size, the reports from Belgium, Russia (including Finland), Italy, Sweden, Spain, Holland, Hungary, Denmark; then we have the present report from Australia and the 16-page report from Roumania.

Professor Carslaw's name is already a familiar one to many readers of the Bulletin, from his elementary texts,* his notable work on the theory of Fourier series and integrals† and his translation of Bonola's Non-Euclidean Geometry.‡

* Plane Trigonometry, an elementary text-book for the higher classes of secondary schools and for colleges. London, 1909. New edition, London, 1915. Key, London, 1914. An Introduction to the Infinitesimal Calculus. Notes for the use of science and engineering students. London, 1905. Second edition, 1912. Reviewed by A. M. Kenyon in this Bulletin, January, 1914, vol. 20, pp. 204–206.

† Introduction to the Theory of Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat. London, 1906. Reviewed by J. E. Wright, in this Bulletin, January, 1909, vol. 15, pp. 196–197.

‡ Non-Euclidean Geometry. A critical and historical study of its development by Robert Bonola. Authorized English translation with additional appendices by H. S. Carslaw with Introduction by F. Enriques. Chicago, 1912. Reviewed by A. Ranum in this Bulletin, October, 1912, vol. 19, pp. 22–23.

Now we have from his pen a clear cut and interesting statement concerning the teaching of mathematics in Australia: in the secondary schools, the technical colleges and schools of mines, the government colleges for the training of teachers, the royal military and naval colleges, and the universities.*

Australia is politically divided into five states (New South Wales, Victoria, Queensland, South Australia, Western Australia) which with the Island of Tasmania form what has been known since 1901 as the commonwealth of Australia. At each of the six capitals, Sydney, Melbourne, Brisbane, Adelaide, Perth, and Hobart, a university supported in part by public funds and in part by private endowments and fees paid by students is established. And while the educational conditions vary greatly in the different states, and not a little in the same state, the universities whatever their status are the "crown of the educational system of which they form a part." The conditions in New South Wales and Victoria, states each with a population of more than one and one half millions and universities founded well over half a century ago, are greatly superior to those in the enormous state of Western Australia with its scattered population of less than 300,000 people and a university which has been in operation little more than a year.

A marked peculiarity of Australian education is that the state "not only controls, but completely and absolutely supports and regulates the system of public education without support from or interference by the localities in which the schools may lie. Australian education tends therefore to be centralized, systematic, and homogeneous; but since local interest is naturally fitful, the external equipment of the schools is usually of an inferior character, while the qualifications of the teachers are distinctly superior. Primary education throughout Australia is free, but secondary is not. The state secondary schools are fewer and somewhat less important than those of a semi-public, endowed, or denominational character."

The organization of secondary education in Australia is passing through a period of active development. But until

---

* Two other recent publications contain a good deal of information about Australian universities: (1) Universities in the Overseas Dominions. Board of Education, Special Reports on Educational Subjects, vol. 25. London, 1912, pp. 116–171, 196–198, 238–269; (2) Congress of the Universities of the British Empire, 1912. Report of Proceedings, London, 1912; also the Year Books for 1914 and 1915.

very recently the chief influence upon the work of the secondary schools has been exerted by the universities, not only through their requirements of matriculation, but also by a system of public examinations taken by pupils of the schools whether they proposed to enter the universities or not. These examinations are similar to the Cambridge and Oxford local examinations. For definiteness let us consider mathematics in New South Wales.

In New South Wales an "intermediate certificate" is given after the successful completion of two years' work (when the pupil is about 16) in the high school. The mathematical course includes arithmetic, mensuration of plane and solid figures, simple numerical trigonometry of the right angled triangle, algebra through simultaneous quadratics, graphs, and Euclid's Elements, Books 1–3.

The third and fourth years' work in the high schools are divided into pass and honor sections and lead to the "leaving certificate." Practically all pupils have to do some mathematical work in these two years, but only those who have shown special aptitude for this study attempt the full course. Indeed, some take only part of the pass course, but all who desire leaving certificates to count as equivalent to university matriculation have to satisfy the examiners in one of the two pass mathematical papers, and thus have to reach a certain standard in algebra, geometry, and trigonometry. There are three higher papers yet, in mathematics: one devoted to geometry and trigonometry; another to algebra, coordinate geometry, and the elements of the differential calculus; and the third to mechanics.

The pass work in algebra includes the following subjects: Logarithms, interest and annuities, graphical illustrations of maxima and minima, binomial theorem for a positive integral index; the additional work for honors includes such subjects as convergence of series, binomial theorem for fractional and negative index, the exponential and logarithmic series, coordinate geometry of the straight line and circle, and a short introduction to the differential calculus.

In geometry the pass work includes the equivalent of Euclid's Elements, Books 4–6; for honors the additional subjects are: Modern geometry, including transversals, nine-point circle, harmonic ranges and pencils, pole and polar, similitude and inversion; solid geometry as in Euclid's Elements,

Book 11, together with theorems relating to the surfaces and volumes of the simpler solid bodies; geometrical conics, including the more important properties of the parabola and ellipse.

Pass trigonometry takes the pupil through the solution of triangles; the honor work includes a fuller treatment, as well as a discussion of DeMoivre's theorem and certain types of series.

Mechanics comes only as one of the higher papers in mathematics. It is intended to be preceded and accompanied by experimental work. The subjects treated are the elements of statics and dynamics, with elementary hydrostatics and atmospheric pressure.

For teachers in secondary schools, public or private, a university degree is an almost indispensable qualification; and not only this, but also special training in the theory and practice of education.

The Teachers' College in Sydney was founded in 1906, mainly for the training of state school teachers. It offers a variety of courses of training varying in length from six months to four years. An ordinary college course—to enter upon which the student must have the leaving certificate or its equivalent— is a two years' course, which qualifies for teaching in the classes of the primary schools. For graduates of the university who wish to prepare as secondary school teachers a one-year course in practical and theoretical education is provided.

Of university mathematical courses which are open to our prospective high school teacher there are three classes, Mathematics I, II, III. Each is divided into three sections: Class A, Class B, and Class C. Candidates for the degree of B.A. or B.Sc. with honors attend the honors' section (Class A) in each year, although it is possible to reach the lowest grade of honors by specially good work in the second section (Class B) in the three years. Here are the programmes for Class A:

Mathematics I (first year): Algebra, geometry, trigonometry, statics and dynamics, analytical geometry of two dimensions, and the elements of the calculus. Those who enter this class are supposed to have honor leaving certificates.

Mathematics II (second year): Infinitesimal calculus, differential equations, spherical trigonometry, analytical statics, particle dynamics, and elementary rigid dynamics.

Mathematics III (third year): Analytical geometry of three dimensions, rigid dynamics, higher analysis, and some applied mathematical subject, e. g., hydrodynamics, sound, the theory of electricity and magnetism.

In addition to these courses a two-year course has recently been instituted in insurance mathematics.

Similar courses are offered in the University of Melbourne and the general educational conditions are very like those in New South Wales. As already remarked, the organization in other states is less advanced. The universities of Australia are staffed by British professors, and thus the mathematical work of the country is fashioned in conformity with much the same ideals as those held by the mother land. Professor Carslaw is at the University of Sydney. Professors Horace Lamb and W. H. Bragg were for many years teachers in the University of Adelaide.

There are several technical colleges in New South Wales, but the work in all of them is, to a great extent, of an elementary character. Higher technical education is available only in the engineering school and mining school of the university. It is hoped that effective cooperation between these schools and colleges may soon be brought about. Schools of mines flourish in Victoria and South Australia. At Victoria mathematical instruction is given in arithmetic, algebra, geometry, plane and spherical trigonometry, analytical geometry of conic sections, differential and integral calculus, and applied mechanics.

There are two institutions in Australia for the early training of the officers of the military and naval forces of the commonwealth, the Royal Australian Military College and the Royal Australian Naval College.

The former is modelled somewhat upon the lines of the United States Military Academy at West Point. Cadets enter at the age of sixteen to nineteen, and receive a training in both military and civil subjects. Mathematics is compulsory for entrance and occupies a prominent position in the first three of the four years of the college course. In the first year 216 hours are given to mathematical lectures and the subjects studied are algebra, geometry, trigonometry, elementary differential and integral calculus, elementary statics and dynamics. In the programme for each subject there is an obligatory and a voluntary section. For example, while a

knowledge of the theory and practical use of logarithms and the use of the slide rule is obligatory, discussion of the exponential theorem of logarithmic series and the calculation of logarithms is optional.

In the second year the hours of instruction in mathematics are the same as in the first year. The subjects of study are: algebra (partial fractions, convergence and divergence of series, simple theorems in probability, etc.); geometry (solid; analytic discussion of straight line and circle; while the voluntary part includes parabola, ellipse and hyperbola); plane and spherical trigonometry (small angles, inverse functions, solution of trigonometrical equations, effect of small errors in surveying and on the trajectory of a bullet, solution of spherical triangles and applications to surveying and astronomy, etc.); astronomy (time, determination of latitude and longitude, correction of instrumental errors, Kepler's laws, etc.); calculus (the voluntary part includes maxima and minima of functions of two variables, approximate numerical evaluation of integrals); dynamics (the voluntary part includes application of the calculus to motion of a particle in a straight line and plane curve, effect of air resistance on a bullet, elementary cases of motion of a rigid body in one plane); statics (center of mass, numerical and graphical calculations relating to tackle, shears, derricks, etc., stresses in frameworks, stresses in a gun, forces in three dimensions, etc.); elementary hydrostatics and hydrodynamics.

In the third year 72 hours of instruction are given in mathematics. The subjects of study are infinitesimal calculus (approximate numerical evaluation of integrals, mean values, etc.); theory of errors; differential equations (ordinary equations of the first order and degree, linear equations, etc.); dynamics; statics.

After satisfactorily completing the four years' course in the college the cadets receive the rank of lieutenant and spend one year in England or India attached to British regiments. They then return to Australia to occupy positions in the permanent military forces. The number of cadets admitted each year is about 40, including 6 who come from New Zealand.

The Naval College was founded for the training of cadet midshipmen who should later join the Australian navy. The training is similar to that at Osborne or Dartmouth in England. Candidates must be thirteen years of age in the year in which they are examined for entrance. The full course lasts for

four years and instruction is given in the following mathematical subjects:—arithmetic, algebra, geometry, plane and spherical trigonometry, analytical geometry, differential and integral calculus.

R. C. ARCHIBALD.

BROWN UNIVERSITY,
PROVIDENCE, R. I.

A CORRECTION.

IN my paper on " Problem Collections in Calculus " in this BULLETIN, June, 1914, volume 20, page 488, line 13, delete " about a dozen signs are wrong and." An unusual form of the equations of an epicycloid led me to consider that in the derivation a slip had been made in sign which required several changes in later work. Since Professor Dingeldey has requested me to rectify my review in this particular, I gladly comply with his wish.

R. C. ARCHIBALD.

SHORTER NOTICE.

Per la biografia di Giovanni Ceva. By GINO LORIA. Reprint from the *Rendiconti* of the Reale Istituto Lombardo di Scienze e Lettere. Pavia, 1915. 3 pp.

STUDENTS of geometry who may have looked into the history of the subject will be interested to know that Professor Gino Loria, of the University of Genoa, has recently been able to fix the date of birth and death of Giovanni Ceva, whose "De lineis rectis se invicem secantibus" appeared at Milan in 1678. Poggendorf gives no dates under the biography of Ceva, but Professor G. Vivanti in the second edition of "Il concetto d'infinitesimo e la sua applicazione alla matematica," in the *Giornale di matematiche,* volumes 38 and 39, quotes M. Pantaleoni as stating that Ceva died in 1734. As a matter of fact, Professor Loria shows, Ceva was born in December, 1647, and died in Mantua on May 13, 1734. It is also interesting to note that Ceva is described in the archives of Mantua as Matematico Cesareo e Commessario Generale dell' Acque di tutto lo Stato, and that he was buried in the Church of Santa Teresa de' Carmelitani Scalzi.

DAVID EUGENE SMITH.

NOTES.

THE twenty-third summer meeting of the American Mathematical Society will be held at Harvard University early in September, 1916. At the eighth colloquium of the Society, held in connection with this meeting, courses of lectures will be delivered as follows: By Professor G. C. EVANS: "Topics from the Theory and Applications of Functionals, including Integral Equations." By Professor OSWALD VEBLEN: "Analysis Situs."

THE opening (September) number of volume 17 of the *Annals of Mathematics* contains the following papers: "A functional equation in the kinetic theory of gases," by T. H. GRONWALL; "Démonstration simplifiée du théorème fondamental de M. Montel sur les familles normales de fonctions," by C. DE LA VALLÉE POUSSIN; "Functions which map the interior of the unit circle upon simple regions," by J. W. ALEXANDER, II; "The iteration of functions of one variable," by A. A. BENNETT.

THE following courses in mathematics are announced for the present winter semester:

TECHNICAL SCHOOL AT DELFT.—By Professor W. J. v. RAAY: Determinants and introduction to the calculus, three hours; Descriptive geometry, three hours; Kinematics and equilibrium, three hours.—By Professor W. A. VERSLUYS: Higher algebra and foundations of the calculus, five hours; Advanced calculus, three hours.—By Professor J. A. SCHOUTEN: Chapters of higher algebra, one hour; Analytic geometry, two hours; Methods of projection, four hours; Curves on given surfaces, four hours.—By Professor J. G. RUTGERS: Plane analytic geometry, two hours; Analytic geometry of space, two hours; Projective methods, four hours; Curves on given surfaces, four hours.—By Professor J. CARDINAAL: General dynamics, two hours; Kinematics, two hours; Introduction to general kinematics, two hours; Advanced kinematics, two hours.—By Professor J. KLOPPER: Introduction to mechanics, one hour; Graphical methods, three hours; Applied mechanics, three hours.

University of Munich.—By Professor F. Lindemann: Analytic mechanics, four hours; Introduction to the theory of ordinary and partial differential equations, four hours; Theory of higher algebraic curves, two hours; Seminar, two hours.—By Professor R. v. Seelinger: Figures of the planets and introduction to the theory of potential, four hours.—By Professor A. Voss: Differential calculus, four hours; Analytic geometry of curves and surfaces, four hours.—By Professor A. Pringsheim: Elements of the theory of functions, five hours.—By Professor A. Sommerfeld: Thermodynamics and the kinetic theory of gases, four hours; Seminar, two hours.—By Professor H. Brunn: Elements of higher mathematics, four hours.—By Dr. F. Hartogs: Descriptive geometry, with exercises, eight hours.—By Dr. F. Böhm: Selected chapters of mathematical statistics, two hours; Elements of the calculus of insurance, two hours; Seminar, two hours.—By Dr. H. Dingler: Elementary mathematics, four hours.—By Dr. A. Rosenthal: Integral calculus, with exercises, five hours.

University of Strassburg.—By Professor F. Schur: Analytic geometry of two and three dimensions, four hours; Selected chapters of differential geometry, two hours; Seminar, two hours.—By Professor G. Faber: Differential and integral calculus, four hours; Elliptic functions, two hours; Seminar, two hours.—By Professor M. Simon: History of mathematics in ancient times, two hours.—By Professor J. Wellstein: Graphical statics, three hours.—By Professor S. Epstein: Analytic treatment of projective geometry, two hours.—By Dr. A. Speiser: Perspective, two hours.

At Brown University Professor R. G. D. Richardson has been made full professor and head of the department of mathematics.

At Dartmouth College Drs. R. D. Beetle and F. M. Morgan have been promoted to assistant professorships of mathematics.

Dr. Bessie I. Miller has been appointed professor and head of the departments of mathematics and physics at Rockford College.

AT the University of North Dakota Professor R. R. HITCHCOCK has been made full professor and head of the department of mathematics.

AT Harvard University Dr. E. V. HUNTINGTON has been promoted from an assistant professorship to an associate professorship of mathematics.

AT the University of Michigan Junior Professors PETER FIELD, L. C. KARPINSKI, and T. R. RUNNING have been promoted to associate professorships of mathematics. Drs. TOMLINSON FORT and T. H. HILDEBRANDT have been promoted from instructorships to assistant professorships of mathematics. Dr. A. L. NELSON has been appointed instructor in mathematics.

AT the University of Illinois Dr. J. B. SHAW has been promoted from an assistant professorship to an associate professorship of mathematics. Dr. L. T. WILSON has been appointed instructor in mathematics.

AT Vassar College Dr. LOUISE D. CUMMINGS has been promoted from an instructorship to an assistant professorship of mathematics. Dr. MARY E. WELLS has been appointed instructor in mathematics.

AT Iowa State College Miss J. T. COLPITTS has been promoted from an assistant professorship to an associate professorship of mathematics.

AT Wesleyan University Dr. J. K. LAMOND has been promoted from an instructorship to an associate professorship of mathematics.

AT Cornell University Dr. JOSEPH SLEPIAN has resigned his instructorship in mathematics, to enter the engineering profession. Mr. H. BETZ has been appointed instructor in mathematics.

DR. NATHAN ALTSHILLER, of the University of Washington, has been appointed instructor in mathematics in the University of Colorado.

THE following appointments to instructorships in mathematics are announced: Dr. R. B. ROBBINS, Sheffield Scientific School; Dr. C. E. WILDER, Pennsylvania State College; Mr. W. L. HART, University of Montana.

NEW PUBLICATIONS.

1. HIGHER MATHEMATICS.

Berger (P.). Die Zerlegung der Lagrange'schen Resolventen für die Kreisteilungsgleichungen in ihre Primfaktoren. Marburg, 1914. 8vo. 42 pp. M. 2.00

Bubnow (N.) und Lezius (J.). Arithmetische Selbständigkeit der europäischen Kultur. Ein Beitrag zur Kulturgeschichte. Berlin, Friedländer, 1915. Gr. 8vo. 8+285 pp. M. 10.00

Emmerich (A.). Aufgaben zu den Grundlehren von den Koordinaten und den Kegelschnitten. Essen, Baedecker, 1915. 64 pp. Geb. M. 1.40

Enriques (F.). Vorlesungen über projective Geometrie. Deutsche Ausgabe von Hermann Fleischer, mit einem Einführungswort von Felix Klein. 2te Auflage. Leipzig, Teubner, 1915. M. 9.00

Fleischer (H.). See Enriques (F.).

Fricke (R.). Analytische Geometrie. (Leitfaden für den mathematischen und technischen Hochschulunterricht.) Leipzig, Teubner, 1915. 8vo. 6+135 pp. Geb. M. 2.80

Gellner (H.). Die Anpassung der Altersgrenzen an den konkreten Fall. Jena, 1914. 8vo. 53 pp.

Hardy (G. H.) and Riesz (M.). The general theory of Dirichlet's series. Cambridge, University Press, 1915. 78 pp. 3s. 6d.

Himstedt (A.). Ueber Polyzonalkurven 4. Ordnung. Nordhausen, 1914. 8vo. 23 pp.

Höppner (W.). Elliptische Koordinaten der Geraden in der Ebene, des Strahles und der Ebene im Bündel. Rostock, 1913. 8vo. 108 pp.+ 5 Tafeln.

Jordan (C.). Cours d'analyse de l'Ecole polytechnique. Tome III: Calcul intégral (équations différentielles). 3e édition. Paris, Gauthier-Villars, 1915. 8vo. Fr. 15.00

Klein (F.). See Enriques (F.).

Koschmieder (Q.). Anwendung der elliptischen Funktionen auf die Bestimmung konjugierter Punkte bei Problemen der Variationsrechnung. Breslau, 1913. 8vo. 86 pp.

Lezius (J.). See Bubnow (N.).

Longley (W. R.). Tables and formulas for solving numerical problems in analytic geometry, calculus and applied mathematics. Revised edition. Boston, Ginn, 1915. 12mo. 6+37 pp. Cloth. $0.50

Merrill (H. A.). Selected topics in college algebra. Privately printed. Norwood, Mass., Norwood Press, 1914. $1.10

Netto (E.) Algebra. (Grundlehren der Mathematik für Studierende und Lehrer, 1ter Teil, 2ter Band.) Leipzig, Teubner, 1915. Gr. 8vo. 12+232 pp. Geb. M. 7.20

Oppermann (A.). Carl Friedrich Gauss. Braunschweig, 1914. 4to. 27 pp.

RIESZ (M.). See HARDY (G. H.).

ROTH (L.). Ueber die singulären Stellen des Haupttangentenkurvensystems einer Fläche. (Diss., Techn. Hochschule, München.) Borna-Leipzig, R. Noske, 1914.

SCHÄDELIN (P.). Eine Regelfläche 4. Grades mit zwei Doppelgeraden und einer Doppelerzeugenden. Bern, 1914. 8vo. 48 pp. M. 1.60

VILLANI (N.). L'equazione di Fermat $x^n + y^n = z^n$, con dimostrazione generale. Lanciano, F. Fanci, 1914. 8vo. 44 pp. L. 3.00

ZISTLER (P.). Rationale Polkurven 4. Ordunung und die Doppeltangententheorie der Kurven 4. Ordnung mit drei Doppelpunkten. Erlangen, 1913. 8vo. 40 pp.+1 Tafel.

II. ELEMENTARY MATHEMATICS.

ABERCROMBIE (J.) et DEGOY (G.). Calculateur commercial, donnant les équivalents en français des prix, poids et mesures anglais et américains et vice versa, avec une méthode simple pour trouver le cubage des caisses en pieds et pouces. Paris, Impr. de la Bourse, 1915. 12mo. 16 pp. Fr. 1.00

CARDAN (G.). See HEFELE (H.).

DEGOY (G.). See ABERCROMBIE (J.).

HARDING (A. M.) and TURNER (J. S.). Plane trigonometry. New York, Putnam, 1915. \$0.90. With tables, \$1.10.

HEFELE (H.). Des Girolamo Cardano von Mailand (Bürgers von Bologna) eigene Lebensbeschreibung. Uebertragen und eingeleitet von H. Hefele. Jena, Diedrichs, 1915. 27+224 pp. Geb. M. 6.50

HÜBNER (M.). Die geschichtliche Entwickelung des Rechenbretts. (Veröffentlichungen des städtischen Schulmuseums zu Breslau.) Breslau, Hirt, 1914. 11 pp. M. 0.10

KEMPTHORNE (W. B.). See WILLIAMS (W. H.).

MÜLLER (E.). See THIEME (H.).

OBERG (E. V.). Elementary algebra. New York, Industrial Press, 1914. Paper. \$0.25

PUNTENNY (M. E.). Two years' work in numbers. Oklahoma City, Puntenny, 1914. \$0.30

RADFORD (E. M.). Mathematical problem papers. Compiled and arranged by E. M. Radford. 2d edition, revised. Cambridge, University Press, 1915. Cr. 8vo. 204 pp. 4s. 6d.

——. Solutions to the problem papers. Cambridge, University Press, 1915. Cr. 8vo. 6+560 pp. 10s. 6d.

RÜEFLI (J.). Kleines Lehrbuch der Stereometrie. Für Mittelschulen. Bern, Francke, 1915. 68 pp. Geb. M. 1.20

SCHUBERT (F.). Grundzüge der ebenen Geometrie. Berlin, 1915. 8vo. 8+223 pp. Geb. M. 2.80

THIEME (H.). Leitfaden der Mathematik für Lyzeen. Bearbeitet von E. Tscharntke. Leipzig, Freitag, 1915. 99 pp. M. 1.30

——. Leitfaden der Mathematik für Oberlyzeen. Bearbeitet von E. Müller. Leipzig, Freytag, 1915. 137 pp. M. 1.70

TSCHARNTKE (E.). See THIEME (H.).

TSCHENTSCHER (A.). See WACKE (R.).

TURNER (J. S.). See HARDING (A. M.).

WACKE (R.) und TSCHENTSCHER (A.). Rechenbuch, bearbeitet nach dem Grundlehrplan für die Volksschulen Gross-Berlins vom 8. 12. 1913. Berlin, Union Deutsche Verlagsgesellschaft, 1914. 11 Hefte. 716 pp. M. 3.85

WILLIAMS (J. H. and K. P.). Plane geometry. Chicago, Lyons and Carnahan, 1915. 8vo. 264 pp. Cloth.

WILLIAMS (K. P.). See WILLIAMS (J. H.).

WILLIAMS (W. H.) and KEMPTHORNE (W. B.). Elementary algebra, complete; shorter course. Chicago, Lyons and Carnahan, 1914. $1.25

——. Second course in algebra. Chicago, Lyons and Carnahan, 1914. $0.80

III. APPLIED MATHEMATICS.

ABRAHAM (M.). Theorie der Elektrizität. (2 Bände.) Band 2: Elektromagnetische Theorie der Strahlung. 3te Auflage. Leipzig, 1914. Gr. 8vo. 10+402 pp. Geb. M. 11.00

AHLFELD (W.). Einfluss von Wind und Luftdruck auf die Höhe des Meeresspiegels. Kiel, 1913. 8vo. 28 pp.

ALT (H.). Zur Theorie der Geschwindigkeits- und Beschleunigungspläne einer komplan bewegten Ebene. (Diss.) Dresden, 1914. Lex. 8vo. 5+73 pp. M. 4.50

ANGERSBACH (—.). Das Relativitätsprinzip in elementarer Behandlung. Weilburg, 1914. 4to. 26 pp.

AUERBACH (F.). Die Physik im Kriege. Allgemein verständliche Darstellung der Grundlagen moderner Kriegstechnik. Jena, 1915. 8vo. 194 pp. M. 3.00

BALL (W. V.). Reminiscences and letters of Sir Robert Ball. London, Cassell, 1915. 16s.

BARTLETT (D. P.). General principles of the method of least squares, with applications. 3d edition. Boston, D. P. Bartlett, 1915. 8vo. 4+142 + 9 pp. Cloth. $2.25

BAUMANN (A.). Der Planet Mars. Zurich, 1913. 8vo. 64 pp. M. 2.50

BIANCHI (E.). Tavole astronomiche per la determinazione del punto. (Istituto centrale aereonautico.) Roma, tip. Unione ed., 1914. 8vo. 15+84+16+24 pp.

BLAIKIE (J.). Peeps at the heavens. New York, Macmillan, 1912. $0.55

BRAHE (T.). Tychonis Brahe Dani opera omnia. Edidit I. L. E. Dreyer. Tome I. Copenhagen, Gyldendalske Boghandel, Nordisk Forlag, 1913. 59+320 pp.

CALENDARIO del R. osservatorio astronomico al collegio romano in Roma. Anno 36 (1915). Roma, tip. Unione ed., 1915. 16mo. 64 pp.

CARDAUN (L.). Messungen am Bogen- und Funkenspektrum des Quecksilbers in internationalen Normalen. Gr. 8vo. 33 pp. M. 1.50

CROMMELIN (A. C.). Star world. Baltimore, Warwick and York, 1915. $0.40

DECHEND (A. v.). Ueber die genaue Messung der Lichtbrechung in Gasen. Heidelberg, 1913. 8vo.

DIELS (H.). Antike Technik. Sechs Vorträge. Leipzig, Teubner, 1914. 6+140 pp. Geb. M. 4.40

DREYER (I. L. E.). See BRAHE (T.).

GADEVOHL (A.). Die Stabilität der Meeresströmungen im Nordatlantischen Ozean südlich 50° N. Br. im Herbst. Kiel, 1913. 19 pp.+1 Karte.

GANS (R.). See WEBER (R. H.).

GRÄTZ (L.). Kurzer Abriss der Elektrizität. 8te Auflage. Stuttgart, 1915. Gr. 8vo. 8+208 pp. Geb. M. 3.50

GUERRIERI (E.). Stelle variabili del tipo di Algol, da osservarsi in Italia durante il 1915: epoche calcolate. Napoli, 1914. 16 mo. 17 pp.

HEIGEL (K. T. v.). Benjamin Thompson, Graf v. Rumford. München, 1915. Lex. 8vo. 30 pp. M. 1.00

HOTOPP (L.). See KECK (W.).

KECK (W.). Vorträge über Mechanik als Grundlage für das Bau- und Maschinenwesen. 3ter Teil: Allgemeine Mechanik. 2te Auflage, bearbeitet von L. Hotopp. Hannover, 1915. M. 11.00

KÖTTER (E.). Ueber den Grenzfall, in welchem ein ebenes Fachwerk von n Knotenpunkten und $2n-3$ Stäben oder ein räumliches Fachwerk von n Knotenpunkten und $3n-6$ Stäben nicht mehr statisch bestimmt ist. Berlin, Verlag der Akademie der Wissenschaften, 1913.

LANGE (M.). Das Schachspiel und seine strategischen Prinzipien. (Aus Natur und Geisteswelt, Band 281.) 2te Auflage. Leipzig, Teubner, 1914. 12mo. 4+108 pp. Geb. M. 1.25

LENZ (K.). Die Rechenmaschinen und das Maschinenrechnen. (Aus Natur und Geisteswelt, Band 490.) Leipzig, Teubner, 1915. 12mo. 6+114 pp. Geb. M. 1.25

MASCHE (W.). Physikalische Uebungen. Ein Leitfaden für die Hand des Schülers. Teil 3: Akustik und Optik. Leipzig, Teubner, 1915. Gr. 8vo. 51 pp. M. 0.80

NÄBAUER (M.). Grundzüge der Geodäsie, mit Einschluss der Ausgleichungsrechnung. (Handbuch der angewandten Mathematik, herausgegeben von H. E. Timerding, 3ter Teil.) Leipzig, Teubner, 1915. 8vo. 16+420 pp. Geb. M. 9.60

OTZEN (R.). Praktische Winke zum Studium der Statik und zur Anwendung ihrer Gesetze. Ein Handbuch für Studierende und praktisch tätige Ingenieure. 2te vermehrte und verbesserte Auflage. Wiesbaden, 1914. M. 5.40

POTINECKE (R.). Fallversuche im luftleeren Raume. 1ter Teil. Magdeburg, 1914. 4to. 19 pp. M. 1.50

SCHLEE (P.). Schülerübungen in der elementaren Astronomie. 2te Auflage. Leipzig, 1915. Gr. 8vo. 15 pp. M. 0.50

SCHREITER (O.). Das exzentrische Schub-Kurbelgetriebe, eine analytische Betrachtung. (Diss.) Rostock, 1914.

SCHULZE (F. A.). See WEBER (R. H.).

SPANUTH (J.). Untersuchung eines automatisch geteilten Kreises. Leipzig, 1913. Gr. 8vo. 41 pp.

TIMERDING (H. E.). See NÄBAUER (M.).

TSCHERNIOVSKY (A.). La mesure des hauts potentiels par l'emploi d'électromètres sous pression. Genève, 1913. 8vo. 32 pp. M. 1.60

VALENTINER (S.). Die Grundlagen der Quantentheorie in elementarer Darstellung. Braunschweig, 1914. M. 2.60

——. Anwendungen der Quantenhypothese in der kinetischen Theorie der festen Körper und der Gase. In elementarer Darstellung. Braunschweig, 1914. M. 2.60

WEBER (R. H.) und GANS (R.). Repertorium der Physik. 1ter Band, 1ter Teil: Mechanik, Elektrizität, Hydrodynamik und Akustik, bearbeitet von R. Gans und F. A. Schulze. Leipzig, Teubner, 1915. 8vo. 434 pp. Cloth. M. 8.00

WEINREICH (W.). Ueber den Temperaturverlauf in stromdurchflossenden Drähten, besonders im Fall von Wechselstrom. Heidelberg, 1913. 8vo. 68 pp. M. 1.80

WEYRAUCH (J. J.). Robert Mayer zur Jahrhundertfeier seiner Geburt. Mit zwei Bildnissen und einer Darstellung der Totenmaske Robert Mayers. Stuttgart, K. Wittwer, 1915.

WIECHERN (W.). Experimentelle und theoretische Untersuchungen über die teilweise Polarisation des im Magnetfelde emittierten Lichtes. Göttingen, 1913. 8vo. 43 pp. + 3 Tafeln. M. 2.00

ZOTH (O.). Ueber die Natur der Mischfarben auf Grund der Undulationshypothese. Braunschweig, 1914. M. 2.80

CONCERNING ABSOLUTELY CONTINUOUS FUNCTIONS.

BY PROFESSOR M. B. PORTER.

In a paper "Sulle funzioni integrali" published in 1905 in the *Atti della R. Accademia delle Scienze di Torino*, Vitali defined an important class of functions of limited variation to which he gave the name of *absolutely* continuous functions. He defines these functions as follows:

Let $F(x)$ be a finite function of the real variable x in an interval (a, b), where $a < b$, and let (α, β) be a partial interval of (a, b), $a \leqq \alpha < \beta \leqq b$. Call $F(\beta) - F(\alpha)$ the *increment* of $F(x)$ in (α, β). Call the sum of such increments, if it is finite and determinate, over a group of distinct (α, β)-intervals, the *increment of $F(x)$ in this group;* then, if for every $\sigma > 0$ there exists a $\mu > 0$ such that the modulus of the increment of $F(x)$ over every group of intervals of sum less than μ is less than σ, then $F(x)$ is said to be absolutely continuous in (a, b). Vitali then shows that $F(x)$ is a continuous function of limited variation, while continuous functions of limited variation are *not* all absolutely continuous, and establishes among others the following important

THEOREM. $F(x) - F(a) \equiv \int_a^x \Lambda F(x)dx$, *where* $\int \Lambda F$ *denotes the Lebesgue integral of one of the derivates of* $F(x)$*; and absolutely continuous functions are the only ones possessing this property.**

Lebesgue had already shown that the derivates of continuous functions of limited variation are summable and that in certain special cases the Lebesgue integral is the primitive function. Vitali's necessary and sufficient condition completes Lebesgue's theory in an important particular and shows that absolutely continuous functions constitute an important generalization of the class of analytic functions, and just as analytic functions can frequently be defined by general descriptive properties it is to be expected that such properties might exist for Vitali's functions. It is the purpose of this paper to show that

* For a proof of this theorem see Vallée Poussin's Cours d'Analyse, Tome 1, § 265, 3d edition.

THEOREM I: *A continuous function of limited variation, whose derivates are infinite only over a denumerable point set E, is absolutely continuous.*

We first remark that in virtue of a theorem of W. H. Young* the set E will always be either denumerable or of the power of the continuum. The inverse theorem is not true, for we shall show, by means of examples, that absolutely continuous functions exist with infinite derivates over any assigned point set of measure zero.

To prove Theorem I, first consider the case where $F(x)$ is monotone increasing. Then†

$$‡\Lambda F(x) \geqq \Lambda \int_a^x \Lambda F(x)dx,$$

so that $\Lambda \int_a^x \Lambda F(x)dx$ is finite whenever $\Lambda F(x)$ is finite. We have now but to apply Vallée Poussin's generalization of Scheeffer's theorem (page 101, ibid.) to see that

$$\int^x \Lambda F(x)dx \equiv F(x) - F(a),$$

which proves that $F(x)$ is absolutely continuous.

To prove the general case we have but to note that

$$\int_a^x |\Lambda F(x)| dx \geqq \int_a^x \Lambda F(x)dx \geqq - \int^x |\Lambda F(x)| dx$$

and again apply Scheeffer's theorem.

As a corollary, we have that if the derivates become infinite over a reducible set, $F(x)$ is absolutely continuous.

We shall now show that no further generalization is possible. To do this consider the function§ $\phi(x)$ defined as follows:

Starting with any null set E and a number c, we take a set of intervals α_{ij} such that

$$\beta_i = \sum_{j=1}^{j=\infty} \alpha_{ij} \leqq \frac{c}{2^i},$$

---

* *Arkiv för Matematik, Astronomi och Fysik*, vol. 1, Stockholm, 1903, or see Hobson's Theory of Functions of a Real Variable, p. 285, for an account of Young's work.

† Cf. Vallée Poussin: Cours d'Analyse, vol. 1, p. 275.

‡ ΛF denotes the upper right-hand derivate of $F(x)$.

§ See Vallée Poussin, loc. cit., vol. 1, p. 100, bottom.

so that

$$\sum_1^\infty \beta_i \leqq c.$$

The points of E are now enclosed in the open intervals α_{ij}, so that each point is inside of an infinite number of intervals, and $\phi(x)$ is defined to be the sum of all the α-intervals or parts thereof which lie to the left of x.

Thus $\phi(x)$ is monotone and can easily be shown to be absolutely continuous as follows:

If $i + j = N$ is chosen sufficiently large, the $\phi_N(x)$ formed for this finite set of intervals will be absolutely continuous and as near as we please to $\phi(x)$ for all values of x. Hence $\phi(x)$ is absolutely continuous.

If the set E is not an *inner limiting* set, the set $E'' = E + \bar{E}$, which lies inside an infinite number of α intervals, will be such a set, and $\phi(x)$ will have an infinite derivative at all the points of E'' and no others. The set E may itself be an inner limiting set, in which case $\bar{E} = 0$.

It would be interesting to determine whether all absolutely continuous functions are of the form

$$F(x) + \phi(x),$$

where $F(x)$ has limited derivates.

AUSTIN, TEXAS.

ON THE REPRESENTATION OF NUMBERS IN THE FORM $x^3 + y^3 + z^3 - 3xyz$.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, August 3, 1915.)

IF by $g(x, y, z)$ we denote the form

$$(1)\quad \begin{aligned} g(x, y, z) &= x^3 + y^3 + z^3 - 3xyz \\ &= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx), \end{aligned}$$

then it is well known that

$$g(x, y, z) \cdot g(u, v, w) = g(xu + yw + zv,\ xv + yu + zw,\ xw + yv + zu).$$

By interchanging the rôles of v and w, we also have

$$g(x, y, z) \cdot g(u, v, w) = g(xu + yv + zw,\ xw + yu + zv,\ xv + yw + zu).$$

Obviously these two representations of the product are identical if $v = w$. Since g is a symmetric function of its arguments it is easy to see that they are identical in each of the following six cases: $v = w$, $v = u$, $w = u$, $x = y$, $x = z$, $y = z$. On the other hand if we assume that the two representations are identical we are led to one of the preceding six equalities. Thus we have the following theorem:*

THEOREM I. *If r, s, t have either of the two sets of values (r_1, s_1, t_1) and (r_2, s_2, t_2), where*

$$(2)\qquad \begin{aligned} r_1 &= xu + yw + zv, & r_2 &= xu + yv + zw,\\ s_1 &= xv + yu + zw, & s_2 &= xw + yu + zv,\\ t_1 &= xw + yv + zu, & t_2 &= xv + yw + zu, \end{aligned}$$

then

$$g(x, y, z) \cdot g(u, v, w) = g(r, s, t).$$

In order that the two expressions $g(r, s, t)$ shall be non-identical it is necessary and sufficient that each of the two sets (x, y, z) and (u, v, w) shall consist of distinct members.

It may be observed that for each set of values (r, s, t) we have

$$r + s + t = (x + y + z)(u + v + w).$$

If a and b are both representable in the form g, then the product ab is representable in the same form, as is seen from the foregoing theorem. The question arises as to whether all the representations of ab are obtained by means of Theorem I from the representations of a and b. That this is to be answered in the negative follows from the simplest examples. Thus it is easy to show that 2 is represented in the form g in only one way, namely, $2 = g(1, 1, 0)$. From this and Theorem I we have $4 = g(2, 1, 1)$, the two sets (r, s, t) being equivalent in this case. But we have also $4 = g(1, 1, -1)$. That is, 4 is capable of a representation in the form g not obtainable by means of Theorem I from the representation of its proper factors.

* The result in this theorem is well known, as we have just pointed out. The remaining theorems in the paper are believed to be new.

From these two representations of 4 it follows that a number may be represented in two ways by the form g and yet these representations not result from writing the product of its factors in two ways in the form g by means of Theorem I. Two other examples illustrating this are afforded by the following relations: $20 = g(3, 1, 1) = g(7, 7, 6)$; $91 = g(6, 4, 3) = g(31, 30, 30)$.

We observe that if the numbers x, y, z, u, v, w in Theorem I are all non-negative then r, s, t are likewise non-negative. This leads us to consider the problem of the representation of numbers in the form g when the arguments are restricted to be non-negative. The fundamental theorem here is the following:

THEOREM II. *Every prime number p other than 3 is representable in one way and in only one way in the form*

$$p = g(x, y, z) \equiv x^3 + y^3 + z^3 - 3xyz, \tag{3}$$

where the arguments x, y, z are restricted to be non-negative.

In order to prove this let us seek to put p in the form

$$p = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx).$$

Since the numbers x, y, z are to be non-negative it is clear that this equation can be satisfied only when

$$x + y + z = p, \quad x^2 + y^2 + z^2 - xy - yz - zx = 1. \tag{4}$$

Without loss of generality we may assume that $x \geqq y \geqq z$, and this we do. Let us write

$$x = u + z, \quad y = v + z.$$

Then u, v, and $u - v$ are non-negative numbers. Equations (4) may now be written

$$3z + u + v = p, \quad u^2 - uv + v^2 = 1. \tag{5}$$

From the latter equation we have $(u - v)^2 + uv = 1$. From this it follows that $u = v = 1$ or $u = 1$, $v = 0$. From the first equation in (5) we see that the former set must be used when p is of the form $3k + 2$ and the latter when p is of the form $3k + 1$, in order that z shall be an integer. In either case u, v, z, and therefore x, y, z, are uniquely determined. Hence the theorem.

Now $g(2, 1, 0) = 9$. From this fact and Theorems I and II

it follows that every positive number is representable in the form $g(x, y, z)$ with non-negative arguments with the possible exception of those of the form $3t$, where t is not divisible by 3. Now, we have

$$g(x, y, z) = (x + y + z)\{(x + y + z)^2 - 3(xy + xz + yz)\}.$$

If the second member of this equation is divisible by 3, so is $x + y + z$, and therefore this second member is divisible by 9 (whatever signs x, y, z may have). Hence the form $g(x, y, z)$ does not contain any number $3t$ where t is an integer prime to 3. Thence we have the following theorem:

THEOREM III. *The positive integers which may be represented in the form $g(x, y, z)$ include all positive integers with the sole exception of those which are divisible by 3 but not by 9. In every case the arguments x, y, z in the representation may be chosen so as to be all non-negative.*

If x, y, z are allowed to be negative it is no longer true that primes are always uniquely represented in the form $g(x, y, z)$. Thus we have $7 = g(3, 2, 2) = g(2, -1, 0)$, $13 = g(5, 4, 4) = g(2, -2, 1)$. Then let us consider more generally the representation of a prime p in the form

$$p = g(x, y, z) = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz).$$

Writing $x = u + z$, $y = v + z$, we have

$$p = (3z + u + v)(u^2 - uv + v^2). \tag{6}$$

Now $4(u^2 - uv + v^2) = (u + v)^2 + 3(u - v)^2$, so that $u^2 - uv + v^2$ is not negative. Hence from (6) it follows that this expression has the value 1 or the value p. Therefore we have to examine the following two cases:

(a) $\qquad u^2 - uv + v^2 = 1, \quad 3z + u + v = p;$

(b) $\qquad u^2 - uv + v^2 = p, \quad 3z + u + v = 1.$

Now the equation $u^2-uv+v^2=1$, or $(u+v)^2+3(u-v)^2=4$, has only the solutions obtained in the proof of Theorem II. Hence case (a) gives rise only to the representation by means of non-negative arguments x, y, z treated in Theorem II.

Let us next consider case (b). We have

$$4p = (u + v)^2 + 3(u - v)^2. \tag{7}$$

Since $p \neq 3$ it follows from this that $p \equiv 1 \bmod 3$, so that p is of the form $6n + 1$. Equation (7) has a solution for every prime p of the form $6n + 1$ such that $u + v \equiv 1 \bmod 3$.* Furthermore $u + v$ and $u - v$ are obviously both odd or both even, so that u and v are themselves integers. From the second equation in (b) it follows now that z, and hence x and y, are integers. Thus we have a representation of p in the desired form $g(x, y, z)$, one at least of the arguments x, y, z being obviously negative. Furthermore it is clear that this representation is unique provided that $4p$ has only one representation

$$4p = a^2 + 3b^2, \quad a > 0, \quad b > 0,$$

in which $a \equiv 1 \bmod 3$, since $u + v$ must have a value congruent to unity modulo 3 in order that z shall be an integer. This latter fact concerning $4p$ we shall now prove. Let $4p$ have the representation

$$4p = \alpha^2 + 3\beta^2, \quad \alpha > 0, \quad \beta > 0.$$

Then we have

$$(8)\quad 16p^2 = (a\alpha + 3b\beta)^2 + 3(a\beta - \alpha b)^2 = (a\alpha - 3b\beta)^2 + 3(a\beta + \alpha b)^2$$

and

$$(9)\qquad 4p(\alpha^2 - a^2) = 3(\alpha b + a\beta)(\alpha b - a\beta).$$

Hence p is a factor of $\alpha b + a\beta$ or of $\alpha b - a\beta$. Suppose that p is a factor of $\alpha b + a\beta$, the complementary factor being s. Then from (8) it follows that p is a factor of $a\alpha - 3b\beta$; let the complementary factor be t. Then from (8) we have

$$16 = t^2 + 3s^2;$$

whence $t = 4$, $s = 0$ or $t = s = 2$. If the former solution is taken, we find from (9) that $a = \alpha$ and hence that the two representations of $4p$ are identical. If we take the latter we have

$$\alpha b + a\beta = 2p, \quad a\alpha - 3b\beta = 2p;$$

whence it follows readily that

$$2\alpha = a + 3b.$$

* See Bachmann's Kreistheilung, pp. 138–141.

Since $a \equiv 1 \bmod 3$ it follows that $\alpha \equiv 2 \bmod 3$. In a similar way one may treat the case when $\alpha b - a\beta$ is divisible by p and with a similar result. Therefore $4p$ can be represented in the form $a^2 + 3b^2$ in only one way provided that a is restricted to be congruent to unity modulo 3.*

We are thus led to the following theorem:

THEOREM IV. *A prime number p of the form $6n + 1$ may be represented in one and in only one way in the form*

$$p = g(x, y, z) \equiv x^3 + y^3 + z^3 - 3xyz$$

where one at least of the arguments x, y, z is negative. No other prime number has such a representation. (Compare Theorem II.)

Let us next consider the representation of p^2 in the form $g(x, y, z)$, p being a prime number different from 3.† Writing $x = u + z$, $y = v + z$, we have

$$p^2 = (3z + u + v)(u^2 - uv + v^2).$$

Since $u^2 - uv + v^2$ cannot be negative it follows that there are three cases to be examined, namely:

(a) $\qquad 3z + u + v = p^2, \qquad u^2 - uv + v^2 = 1;$

(b) $\qquad 3z + u + v = p, \qquad u^2 - uv + v^2 = p;$

(c) $\qquad 3z + u + v = 1, \qquad u^2 - uv + v^2 = p^2.$

These may be treated by the methods already employed. We take up the cases in order.

The second equation in (a) has the two solutions $u = v = 1$; $u = 1, v = 0$, and no others (if we take $u \geq v$, as we may without loss of generality). Since z must be integral it follows from the first equation in (a) that we must take $u = 1, v = 0$. We are thus led to the following conclusion:

There is a unique representation of p^2 ($p \neq 3$) in the form $g(x, y, z)$ subject to the condition $x + y + z = p^2$.

In case (b) it is easy to show from the second equation that p is of the form $6n + 1$. Proceeding as in the proof of Theorem

---

* As a corollary of this argument we have the following result:

If p is a prime number of the form $6n + 1$ then $4p$ can be represented in two and in only two ways in the form $a^2 + 3b^2$, a and b being positive, and in one of these ways a is congruent to 1 and in the other a is congruent to 2 modulo 3.

† For the excluded case we have $9 = g(2, 1, 0)$.

IV, we find that there is a unique solution of equations (b) subject to the condition that z is integral. We thus conclude:

In order that p^2 ($p \neq 3$) *shall be representable in the form* $g(x, y, z)$, *with the condition* $x + y + z = p$, *it is necessary and sufficient that* p *be of the form* $6n + 1$ *and this representation, when it exists, is unique.*

In case (c) the second equation has the obvious solution $u = v = p$. This solution will yield integral z only when p has the form $3k + 2$. The solution is unique for such p since it follows from the theory of binary quadratic forms that such a prime power p^2 can be represented in the form $u^2 - uv + v^2$ only when $u = v = p$ or $u = p, v = 0$, the latter solution giving z non-integral in the present case. If p is of the form $3k + 1$ then the second equation in (c) has the solution $u = p$, $v = 0$; this gives rise to integral z and hence to a representation of the kind sought. The representation in this case is not necessarily unique, since the second equation in (c) may have a second solution giving rise to integral z. We have the following result:

The prime power p^2 ($p \neq 3$) *can be represented in the form* $g(x, y, z)$ *subject to the condition* $x + y + z = 1$.

UNIVERSITY OF ILLINOIS.

ON THE LINEAR CONTINUUM.

BY DR. ROBERT L. MOORE.

(Read before the American Mathematical Society, April 24, 1915.)

§ 1. *Introduction.*

IN the *Annals of Mathematics,* volume 16 (1915), pages 123–133, I proposed a set G of eight axioms for the linear continuum in terms of *point* and *limit.* Betweenness was defined,* and it was stated that the set G is categorical with respect to *point* and *the thus defined betweenness.*† In the present paper it is shown that, although this statement is true, nevertheless

* See Definition 3, loc. cit., p. 125.

† This statement, which is proved in the present paper, implies that if K is any statement in terms of point and betweenness, then either it follows from Axioms 1–8 and Definition 3 that K is true or it follows from Axioms 1–8 and Definition 3 that K is false.

G is not *absolutely* categorical,* that is to say it is not categorical with respect to *point* and *limit,* the undefined symbols in terms of which it is stated.

An absolutely categorical set is obtained if Axiom 5 is replaced by the following axiom.

AXIOM 5′. *If r_1 and r_2 are two mutually exclusive, non-complementary rays,† then every infinite set of points lying in $S - (r_1 + r_2)$ has at least one limit point.*

§ 2. *On the Non-Categoricity of the Set G.*

That G is not categorical is shown by the existence of the following examples E and $E_{5'}$. The letter K will be used to denote the statement that the point P is a *limit point* of the point set M whenever every segment containing P contains at least one point of M distinct from P.

$E_{5'}$. Let the space S be an ordinary linear continuum $(0 < x < 1)$ but interpret the statement that P is a *limit point* of M to mean that P is a limit point in the ordinary sense of a rational subset of M. Here Axioms 1–8 are satisfied‡ but statement K is false.

E. Let S be an ordinary linear continuum and let *limit*

* For definition of categoricity (in the absolute sense) see O. Veblen. "A system of axioms for geometry," *Transactions of the American Mathematical Society*, vol. 5 (1904), pp. 343–384.

† If P is a point and S is the set of all points, and $S - P = S_P' + S_P''$, where S_P' and S_P'' are mutually exclusive connected point sets neither of which contains a limit point of the other one, then S_P' and S_P'' are called *rays.* If B is a point of the ray S_P' then the ray S_P' is denoted by PB. The ray S_P' is said to be complementary to (or the complement of) the ray S_P''.

‡ That Axiom 5 is satisfied in this example may be proved as follows. In this proof the phrase "limit point" (unitalicized) has its ordinary meaning while "*limit point*" (in italics) is to be interpreted as defined in E_5'.

Suppose that $S = K_1 + K_2$ where K_1 and K_2 are mutually exclusive point sets. There are two cases to be considered.

Case I. Suppose K_1 contains no rational subset. Then K_2 contains the set of all rational points. But every point of S is a limit point of this set. Hence every point of K_1 is a *limit point* of K_2.

Case II. Suppose $K_1 = R_1 + I_1$ and $K_2 = R_2 + I_2$, where R_1 and R_2 are composed entirely of rational points, while I_k $(k = 1, 2)$ either is vacuous or is composed entirely of irrational points. Suppose K_1 contains no limit point of R_2 and K_2 contains no limit point of R_1. Then one of the point sets I_1 and I_2 must contain a limit point of the other one. Suppose I_1 contains a limit point of I_2. Since every point of I_2 is a limit point of $R_1 + R_2$ but not of R_1, therefore every point of I_2 is a limit point of R_2. It follows that I_1 contains a limit point of R_2 and therefore K_1 contains a *limit point* of K_2.

point have its usual significance. In this example also Axioms 1–8 are all satisfied. But here the statement K is true.

From the existence of these two examples it is clear that neither K nor its contradictory is a consequence of Axioms 1–8. Hence this system of axioms is not absolutely categorical.

§ 3. *Consequences of Axioms 1–4, 6, 7.*

Theorem A. *No point is a limit point of a finite set of points.*

Theorem A is a consequence of Axioms 2 and 3.

Theorem B. *Every ray contains infinitely many points.*

Theorem B is a consequence of Theorem A and Theorem 1.*

Theorem C. *Every ray contains an infinite set of points that has no limit point.*

Proof. By Axiom 7 there exists a countable set of points R such that every point either belongs to R or is a limit point of R. If the ray AB did not contain infinitely many points of R, then, by Theorem B, Axiom 2 and Theorem A, AB would contain a limit point of AB', its complement. But this is contrary to Definition 2. Hence AB and R contain infinitely many points in common. Let P_1, P_2, P_3 be the set of all such common points. By Theorem 11 there exists a point X_1 such that AP_1X_1. There exists K_1 such that AX_1K_1. By Theorem 2, AX_1 contains X_1K_1. But AX_1 is the same as AB. Thus AB contains X_1K_1. But, by Theorem 4, X_1P_1A. Therefore P_1 is on X_1A. Consequently P_1 is not on X_1K_1. Thus the ray X_1K_1 lies in AB but does not contain P_1. Similarly there exists a ray X_2K_2 lying in X_1K_1 (and therefore in AB) and not containing P_2. Continue this process, thus obtaining two sequences of points $X_1, X_2, \cdots$ and $K_1, K_2, \cdots$ such that AB contains X_nK_n, X_nK_n contains $X_{n+1}K_{n+1}$ and X_nK_n contains no point of the set $P_1, P_2, \cdots, P_n$. Suppose the infinite set of points $X_1, X_2, X_3, \cdots$ has a limit point X. The points $X_{n+1}, X_{n+2}, X_{n+3}, \cdots$ all lie on X_nK_n. Hence for every n, X lies on X_nK_n. Now A is not on X_nK_n. Hence it is not on X_nX. Therefore AX_nX is true for every n. Hence XA contains every X_n. But X_nA is the complement of X_nK_n, and therefore contains $P_1, P_2, P_3, \cdots, P_n$. Furthermore XA contains X_nA. Therefore XA contains all the points

---

* Arabic numerals are used for theorems and definitions contained in my paper "The linear continuum in terms of point and limit," loc. cit.

$P_1, P_2, \cdots, P_n$. But there exists a point Y such that AXY. The rays XY and XA are complementary. Therefore XY contains no P_n. But XY is a subset of AB. Therefore XY contains no point of R. Hence Y is not a limit point of R. Thus the supposition that the set of points $X_1, X_2, \cdots$ has a limit point leads to a contradiction.

§ 4. *Consequences of Axioms 1–4, 5′, 6, 7.*

THEOREM D.* *There do not exist three mutually exclusive rays.*

Theorem D is a consequence of Axiom 5′ and Theorem C.

THEOREM E. *If P is a limit point of M then every segment containing P contains at least one point of M distinct from P.*

Proof. Let AB denote a segment† containing P. There exist points C and D such that ABC and BAD. The ray BC is the complement of BA, while AD is the complement of AB. If the point X does not belong to the segment AB, then, by Theorems 14 and 4 and Definition 3, X is not common to the rays AB and BA. Hence if no point of M except P is in the segment AB then $M = M_1 + M_2$, where no point of M_1 except P is in AB and no point of M_2 is in BA. But P is in both AB and BA. Hence P is a limit point of neither M_1 nor M_2. Therefore, by Axiom 2, P is not a limit point of M. But this is contrary to hypothesis.

THEOREM F. *There exists a countable, everywhere dense‡ set of points.*

Theorem F is a consequence of Axiom 7 and Theorem E.

It follows§ from Theorems 4–14 and Theorem F that the set of Axioms 1–7|| is categorical with respect to *point* and *betweenness* as defined in Definition 3.

THEOREM G. *If every segment containing P contains at least one point of M distinct from P then P is a limit point of M.*

Proof. Between S and the linear continuum $(0 < x < 1)$ there is a one-to-one reciprocal correspondence that *preserves*

---

* See Axiom 5, loc. cit., p. 126.

† The *segment* AB is the set of all points $[X]$ such that AXB.

‡ A set of points M is said to be everywhere dense if every segment contains a point of M.

§ See G. Cantor, "Zur Begründung der transfiniten Mengenlehre, I," *Mathematische Annalen*, vol. 46 (1895), p. 510.

|| It is to be noted that Theorems E and F are both consequences of Axioms 1–7 as well as of Axioms 1–4, 5′, 6, 7.

order. It follows that M contains an infinite set of points $P_1, P_2, P_3, \cdots$ such that for every segment τ containing P there exists n such that $P_{n+1}, P_{n+2}, P_{n+3}, \cdots$ all lie in τ. It follows, with the help of Axiom 5′, that the set of points $P_1, P_2, P_3, \cdots$ has at least one limit point O. Suppose that O is distinct from P. Then there exist points A, B, and C in the order $APBOC$. There exists n such that $P_{n+1}, P_{n+2}, P_{n+3}, \cdots$ all lie in the segment AB. Hence not more than n points of the set $P_1 + P_2 + P_3 \cdots$ lie in the segment BC. Therefore, by Theorem E, Axiom 2 and Theorem A, O is not a limit point of $P_1 + P_2 + P_3 \cdots$. Thus the supposition that P is distinct from O leads to a contradiction. It follows that P is a limit point of $P_1 + P_2 + P_3 + \cdots$ and therefore of M.

§ 6. *Conclusion.*

THEOREM H. *The set of Axioms 1–4, 5′, 6, 7 is an absolutely categorical set of axioms for the linear continuum.*

Proof. It has been shown that this set of axioms is categorical with respect to point and betweenness as defined in Definition 3. But every statement in terms of *point* and *limit point* of a point set is,* in the presence of these axioms and Definition 3, equivalent to a statement in terms of *point* and *betweenness*. It follows that the set of Axioms 1–4, 5′, 6, 7 is categorical with respect to *point* and *limit point* of a point set.

That, in the set of Axioms 1–4, 5′, 6, 7, Axioms 2, 3, 4, 5′, and 6 are independent is shown by Examples E_2–E_6 of my paper in the *Annals*. That 1 and 7 are independent in this set is shown by the following examples, E_1 and E_7.

E_1. S is an ordinary linear continuum. The point P is a *limit point* of the point set M if and only if P is a limit point of M in the usual sense but M is not the set of all points.

E_7.† S is the set of all real number pairs (x, y) such that $0 < x < 1$ and $0 \leqq y \leqq 1$. The point (x_1, y_1) is a *limit point* of the point set M if, and only if, it is true that corresponding to each preassigned positive number ϵ there exists, in the set

---

* See Theorems E and G.

† The example E_7 was constructed with the assistance of an example given by Veblen in connection with his postulate of uniformity. Cf. O. Veblen, "Definition in terms of order alone in the linear continuum and in well-ordered sets," *Transactions of the American Mathematical Society*, vol. 6 (1905), p. 169.

M, a point (x, y), distinct from (x_1, y_1), such that 1) $|x - x_1| < \epsilon$, 2) $x - x_1 = 0$ and $|y - y_1| < \epsilon$, in case y_1 is distinct from 0 and from 1, 3) $x \leqq x_1$ if $y_1 = 0$, 4) $x \geqq x_1$ if $y_1 = 1$.

UNIVERSITY OF PENNSYLVANIA.

A PROBLEM IN THE KINEMATICS OF A RIGID BODY.

BY PROFESSOR PETER FIELD.

THE problem of finding the acceleration of any point in a rigid body when the accelerations of three points are given, and incidentally of finding what is by this means determined regarding the velocities, has received but little attention. A theorem due to Burmeister solves the problem of finding the acceleration of any point in the plane of the three points whose accelerations are given. The theorem states: "If at four coplanar points P_1, P_2, P_3, P_4 the accelerations be drawn, their extremities lie in a plane and form a quadrilateral which is affine with the quadrilateral formed by the four points."

R. Mehmke* and J. Petersen† have considered the general case, but their results do not agree, owing to an oversight in Petersen's treatment. While their work is independent, the proof in both cases depends directly on the fact that when the distance between two points is constant the projections of their velocities on their joining line are equal and the projections of their accelerations on this line differ by $\omega^2 l$, l being the distance between the two points and ω the angular velocity of the line. The purpose of this paper is to show that the problem can be solved very simply by using the expressions for the accelerations which are ordinarily given in text books on mechanics, and by this method the kinematical meaning of the solution is also evident.

Let there be given the accelerations at three points. It is proposed to find what can be determined regarding the kinematical state of the body at the given instant. As the acceleration at any point in the plane of the three points can be

---

* Festschrift zur Feier des 50jährigen Bestehens der technischen Hochschule Darmstadt, page 77.

† Kinematik, page 46.

found by Burmeister's theorem, it is no restriction to assume that the points (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2) whose accelerations are given have at the given instant the coordinates $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$.

The general formulas for the components $(\ddot{x}, \ddot{y}, \ddot{z})$ along the fixed axes of the acceleration of any point (x, y, z) of the moving body may be written*

$$\ddot{x} = \ddot{x}_0 + \omega_x(\omega_x x + \omega_y y + \omega_z z) - \omega^2 x + \dot{\omega}_y z - \dot{\omega}_z y,$$

$$\ddot{y} = \ddot{y}_0 + \omega_y(\omega_x x + \omega_y y + \omega_z z) - \omega^2 y + \dot{\omega}_z x - \dot{\omega}_x z,$$

$$\ddot{z} = \ddot{z}_0 + \omega_z(\omega_x x + \omega_y y + \omega_z z) - \omega^2 z + \dot{\omega}_x y - \dot{\omega}_y x.$$

In these equations ω is the angular velocity of the body, $\dot{\omega}$ the angular acceleration, and $(\ddot{x}_0, \ddot{y}_0, \ddot{z}_0)$ is the acceleration of that point in the body which at the given moment coincides with the origin of the axes of reference. These formulas applied to the points (x_1, y_1, z_1) and (x_2, y_2, z_2) give

$$\ddot{x}_1 = \ddot{x}_0 - (\omega_y^2 + \omega_z^2), \quad \ddot{y}_1 = \ddot{y}_0 + \omega_x\omega_y + \dot{\omega}_z, \quad \ddot{z}_1 = \ddot{z}_0 + \omega_x\omega_z - \dot{\omega}_y,$$

$$\ddot{x}_2 = \ddot{x}_0 + \omega_x\omega_y - \dot{\omega}_z, \quad \ddot{y}_2 = \ddot{y}_0 - (\omega_x^2 + \omega_z^2), \quad \ddot{z}_2 = \ddot{z}_0 + \omega_z\omega_y + \dot{\omega}_x.$$

These equations determine ω and $\dot{\omega}$ when the accelerations of the three points are given.

It is more convenient to say that one of the possible solutions gives p, q, r as the components of ω and l, m, n as the components of $\dot{\omega}$. Any other solution must satisfy the equations

$$q^2 + r^2 = \omega_y^2 + \omega_z^2, \quad pq + n = \omega_x\omega_y + \dot{\omega}_z, \quad pr - m = \omega_x\omega_z - \dot{\omega}_y,$$

$$pq - n = \omega_x\omega_y - \dot{\omega}_z, \quad p^2 + r^2 = \omega_x^2 + \omega_z^2, \quad qr + l = \omega_z\omega_y + \dot{\omega}_x.$$

It follows that the components of ω and $\dot{\omega}$ may be any one of the following:

I. (p, q, r), (l, m, n),

II. $(-p, -q, -r)$, (l, m, n),

III. $(p, q, -r)$, $(l + 2qr, m - 2pr, n)$,

IV. $(-p, -q, r)$, $(l + 2qr, m - 2pr, n)$.

This shows that the absolute value of ω is determined, but

* See for instance Ziwet and Field, Introduction to Analytical Mechanics, p. 107.

the direction of the axis of spin is not. The two axes lie in a plane perpendicular to the xy plane and they make equal angles with the xy plane. The two values of the angular acceleration have the same projection along the z axis but their projections on the xy plane differ by a vector which is perpendicular to the projection of ω on this plane and equal to $2r$ times this projection. [It might be more convenient to view the two values of $\dot{\omega}$ as having the components $l + qr \pm qr, \quad m - pr \pm pr, n.$]

For I or II the components of the acceleration of any point (x, y, z) are

$$
\begin{aligned}
\ddot{x} &= \ddot{x}_0 - (q^2 + r^2)x + (pq - n)y + (pr + m)z,\\
\ddot{y} &= \ddot{y}_0 + (pq + n)x - (p^2 + r^2)y + (qr - l)z,\\
\ddot{z} &= \ddot{z}_0 + (pr - m)x + (qr + l)y - (p^2 + q^2)z;
\end{aligned}
\tag{1}
$$

for III or IV they are

$$
\begin{aligned}
\ddot{x} &= \ddot{x}_0 - (q^2 + r^2)x + (pq - n)y + (m - 3pr)z,\\
\ddot{y} &= \ddot{y}_0 + (pq + n)x - (p^2 + r^2)y - (l + 3qr)z,\\
\ddot{z} &= \ddot{z}_0 + (pr - m)x + (l + qr)y - (p^2 + q^2)z.
\end{aligned}
\tag{2}
$$

It is no restriction to take the axis of rotation in the yz plane, i. e., $p = 0$. In place of (1) and (2) we then have (1′) and (2′)

$$
\begin{aligned}
\ddot{x} &= \ddot{x}_0 - (q^2 + r^2)x - ny + mz,\\
\ddot{y} &= \ddot{y}_0 + nx - r^2y + (qr - l)z,\\
\ddot{z} &= \ddot{z}_0 - mx + (qr + l)y - q^2z,
\end{aligned}
\tag{1′}
$$

and

$$
\begin{aligned}
\ddot{x} &= \ddot{x}_0 - (q^2 + r^2)x - ny + mz,\\
\ddot{y} &= \ddot{y}_0 + nx - r^2y - (l + 3qr)z,\\
\ddot{z} &= \ddot{z}_0 - mx + (l + qr)y - q^2z.
\end{aligned}
\tag{2′}
$$

These equations show that there are two possible values for the acceleration of any point not in the xy plane. These values become equal if either q or r is equal to zero; i. e., if the axis of spin is either parallel or perpendicular to the plane of the three points. If neither q nor r is equal to zero, the center of acceleration is different for the two cases unless it should happen to lie in the xy plane.

Summary.—When the accelerations of three points in a rigid body are given, the acceleration of any point in the plane of the given points is determined uniquely. The acceleration of a point not in the plane of the given points is in general two valued. Moreover, there are in general four sets of values of ω and $\dot{\omega}$ which give the same values for the accelerations of the points in a given plane. For a given value of ω there can be determined the value of the spin for the line joining a given pair of points and hence the relative velocity of the two points can be found.

University of Michigan,
September, 1915.

JULES HENRI POINCARÉ.

Enquête de "l'Enseignement Mathématique" sur la méthode de travail des mathématiciens. Publié par H. Fehr avec la collaboration de T. Flournoy et E. Claparède. Deuxième édition conforme à la première suivie d'une *Note sur l'invention mathématique* par H. Poincaré. Paris, Gauthier-Villars, et Genève, George, 1912. 8vo. 8+137 pages. Price 5 francs.

Notice sur Henri Poincaré. Par E. Lebon. Paris, Hermann, 1913. 8vo. xlviii pages. Price 2 francs.

Savants du Jour: Henri Poincaré, Biographie, Bibliographie analytique des écrits. Seconde édition entièrement refondue. Par E. Lebon. Paris, Gauthier-Villars, 1912. Royal 8vo. 112 pages. Price 7 francs.

It was in the latter part of 1900 that M. E. Maillet wrote as follows:* "Messieurs les Rédacteurs, Il y aurait, ce me semble, une tentative à faire, pour laquelle l'*Enseignement Mathématique* est à mon avis tout à fait désigné, et dont le succès pourrait rendre de bien grands services aux jeunes mathématiciens. Elle consisterait à ouvrir une sorte d'enquête auprès de savants connus; il s'agirait d'obtenir de chacun d'eux quelques renseignements personnels sur sa méthode de travail et de recherche, ses habitudes, l'hygiène générale qu'il juge la plus propre à faciliter son travail intellectuel, la manière de conduire le plus efficacement ses lectures et d'en tirer le meilleur parti, etc., etc. Je me borne ici à indiquer les grandes

* *L'Enseignement Mathém ique*, 1901, tome 3, p. 58.

lignes, tout en reconnaissant que, si mon idée était mise en exécution, il y aurait lieu de pénétrer un peu plus dans le détail."

As an outcome of the letter of which this is the first paragraph, thirty questions were formulated and published,* and copies were distributed among mathematicians at the international congresses in Heidelberg and St. Louis; a certain number of copies were also addressed to savants of different countries. By 1905 a considerable body of material had been collected with reference to the questionnaire, and during the years 1905–1908† this material was published in suitable synoptic form. The whole was collected and first issued as a pamphlet in 1909.

The results of the pamphlet are deduced from documentary evidence of more than a hundred mathematicians who are, for the most part, living; about a score preferred to have their testimony published anonymously. In the inquiry America is represented by statements of such men as Professors Coolidge, E. W. Davis, Dickson, Emch, F. R. Moulton, Rietz, Snyder, J. W. Young, and Mr. Escott.

Here are some illustrations of the questionnaire and of the responses:

Question 1 (a): At what period of your recollection and under what circumstances did the taste for mathematics take possession of you? To this question 93 replies were received; 35 placed the period before 10 years of age, 43 from 11 to 15 years of age, 11 from 16 to 18 years, 3 from 19 to 20 years, and 1 at 26 years of age. M. Lecat reported, "At $3\frac{1}{2}$ my attention was strongly fixed on the idea of number;" Professor Dickson replied, "At the age of 5. . . . At 12 years of age I had decided to pursue mathematical study." The well-known facts concerning such infant mathematical prodigies as Clairaut, Gauss, Ampère, and Bertrand are also recalled. While Steiner showed early aptitude for oral calculation and astronomy, it was not till 18 years of age that he even learned to write.

Question 6: Have you sought to learn the genesis of the truths, discovered by you, to which you attach the greatest importance? *Question 7:* To what extent, in your opinion, do chance and inspiration play a part in mathematical discovery? Is this part always as large as it appears? *Question*

* *L'Enseignement Mathématique*, 1902, tome 4, pp. 208–211; tome 6, 1904, pp. 376–378, 481.

† *L'Enseignement Mathématique.* A complete list of references is given in tome 10, 1908, p. 172.

8(a): Have you sometimes remarked that discoveries or solutions in subjects entirely foreign to your researches of the moment have presented themselves to you, and that they corresponded to research formerly fruitless? *Question 8(b)*: Have you ever calculated or solved problems in a dream? Or have solutions and discoveries appeared to you in their completeness on waking up in the morning, when they were vainly sought in the evening or days previously? *Question 9:* Do you estimate that your principal discoveries have been the result of a voluntary endeavor directed in a definite way, or have they occurred to you spontaneously, so to speak?

The replies are summed up as follows (pages 47–48): "Mathematical discoveries—small or great, and whatever their content (new subjects of research, divination of methods or of lines to follow, presentiments of truths and of solutions not yet demonstrated, etc.)—are never born by spontaneous generation. They always presuppose a ground sown with preliminary knowledge and well prepared by work both conscious and subconscious. On the other hand every discovery by its very novelty and originality contrasts forcibly with such a statement and appears all the more surprising when it leaps forth unexpectedly from a very prolonged incubation. We learn, then, that according to the cases and the individuals it is sometimes the unlooked for character, and sometimes the dependence on previous work which strikes the author most when he reflects retrospectively. Whence so many varieties of estimate and equal truth of these two celebrated aphorisms, contradictory in appearance, but expressing the two aspects indissolubly bound together, although of relief often very unequal, of the same process: le génie, c'est l'inspiration; le génie, c'est une longe patience."

In connection with Question 8 (*b*), reference might have been made to the composition of Maria Agnesi's Instituzioni Analitiche (1748). "To this difficult task," H. J. Mozans writes,* "she devoted ten years of arduous and uninterrupted labor. And if we may credit her biographer, she consecrated the nights as well as the days to her herculean undertaking. For frequently, after working in vain on a difficult problem during the day, she was known to bound from her bed during the night while sound asleep, like a somnambulist, make her way through a long suite of rooms to her study, where she wrote

* Woman in Science, New York and London, 1913, pp. 144–145.

out the solution of the problem and then returned to bed. The following morning, on returning to her desk, she found to her great surprise, that while asleep she had fully solved the problem which had been the subject of her meditations during the day and of her dreams during the night."

Poincaré's " Note " on Mathematical Discovery is of extraordinary interest. It was delivered, in a conférence at the Institut général psychologique in May, 1908, and was first published in the Institut's *Bulletin* for June of the same year. A few months later it was included in the volume entitled Science et Méthode.* The opening sentences are as follows:

" The genesis of mathematical discovery is a problem which must inspire the psychologist with the keenest interest. For this is the process in which the human mind seems to borrow least from the exterior world, in which it acts, or appears to act, only by itself and on itself, so that by studying the process of geometric thought we may hope to arrive at what is most essential in the human mind.

" This has long been understood, and a few months ago a review called *l'Enseignement Mathématique* . . . instituted an inquiry into the habits of mind and methods of work of different mathematicians. I had outlined the principal features of this article when the results of the inquiry were published, so that I have hardly been able to make any use of them, and I will content myself with saying that the majority of the evidence confirms my conclusions."

The article deals mainly with just such topics as are suggested by questions quoted above. In the first of its five sections the author considers the following questions and answers: How does it happen that there are people who do not understand mathematics? How is error possible in mathematics? What is mathematical discovery? Poincaré points out that, especially in connection with discovery, intuition of mathematical order is fundamental. "A mathematical demonstration is not a simple juxtaposition of syllogisms; it consists of syllogisms *placed in a certain order*, and the order in which these elements are placed is much more important than the elements themselves. If I have the feeling, the intuition so to speak, of this order so that I can perceive the whole of

* Cf., for example, the English translation by Francis Maitland with a preface by B. Russell, London, 1914, pp. 46–63.

the argument at a glance, I need no longer be afraid of forgetting one of the elements; each of them will place itself naturally in the position prepared for it, without my having to make any effort of memory." "Discovery is discernment, selection," and selection is made by the intuition for order.

"But what I have said up to now," Poincaré remarks in commencing the second section, "is only what can be observed or inferred by reading the works of geometers, provided they are read with some reflection. It is time to penetrate further, and to see what happens in the very soul of the mathematician." And Poincaré recounts his recollections of how he wrote his first treatise on Fuchsian functions. Some of the underlying ideas are developed in the following section. Then there is a summing up in section four. "The result of all that precedes is to show that the unconscious ego or, as it is called, the subliminal ego, plays a most important part in mathematical discovery."

Poincaré considers as a first hypothesis: "The subliminal ego is in no way inferior to the conscious ego; it is not purely automatic; it is capable of discernment; it has tact and lightness of touch; it can select, and it can divine. More than that, it can divine better than the conscious ego, since it succeeds where the latter fails. In a word, is not the subliminal ego superior to the conscious ego?" Poincaré explains why he is loth to give an affirmative answer to this question. He prefers rather to consider: "Of the very large number of combinations which the subliminal ego blindly forms, almost all are without interest and without utility. But, for that very reason, they are without action on the esthetic sensibility; the consciousness will never know them. A few only are harmonious, and consequently at once useful and beautiful, and they will be capable of affecting the geometer's special sensibility I have been speaking of; which, once aroused, will direct our attention on them, and will thus give them the opportunity of becoming conscious." "The conscious ego is strictly limited." Limitations and characteristics of the subliminal ego are considered in the last section.

Such in barest outline are some of the thoughts.

M. Lebon's "Notice" is a reprint in pamphlet form of the introduction to the second edition of Poincaré's Hypothèses Cosmogoniques. The first half of the memoir is "Sur la vie

de Henri Poincaré," the latter, "Sur les travaux scientifiques de Henri Poincaré" and here M. Lebon tells us that his aim has only been to "signaler les beaux résultats des recherches originales de Henri Poincaré et les démonstrations rigoureuses qu'il donne, en insistant surtout sur les idées directrices de ses profondes études. D'une part, je souhaite avoir réussi à indiquer nettement comment son esprit fécond et universel est parvenu à élucider les théories obscures, à étendre le domaine des théories naissantes ou à en signaler les défauts. D'autre part, j'espère donner, à ceux qui ont en vue de faire progresser la Science, l'idée de lire avec attention tous ses écrits, qui leur montreront soit la meilleure marche à suivre, soit les points qu'il jugeait dignes d'être encore approfondis."

The first part of the "Notice" is especially interesting. As the topic is less dwelt upon in memorial sketches, a few quotations may be appropriately given in illustration of the whole.

"Henri Poincaré possessed, to a high degree, intuition of mathematical nature. At the Nancy lycée his comrades were struck by it. M. Paul Appell, his codisciple in Mathématiques Spéciales, affirms that he already had 'le don génial d'appercevoir intuitivement, avec le détail particulier de chaque question, l'idée générale dont elle procède, et la place qu'elle occupe dans l'ensemble.' From his first year at the lycée, Henri Poincaré had a method of working all his own. He had to force himself to sit at a study table, and neither noise nor conversation disturbed the working of his mind. To fix his thought on a subject there was no need for auxiliary material; it sufficed that a logical thread pervaded it, in order that it could not escape him. . . .

"Mathematician, philosopher, poet, artist, Henri Poincaré had to be also a great writer. His only aim was to express his thought with all his sincerity and to communicate to his readers his emotions and noblest enthusiasms. He wrote with a dash of the pen, for his ideas were of such a delicate nicety, his thoughts so excessively active, they almost always found immediately their perfect expression. . . .

"The style, infinitely supple and varied, is now the style of the savant, then that of the scholar or of the poet: it is also that of a writer truly French and of the line of the Montaignes, of the Molières, and of the Pascals. Elegant, simple, limpid, of great conciseness, this style abounds in amusing sallies, in an irony

occasionally cutting, but these sallies were aimed at ridiculous things, never at persons. It abounds also in pleasing and picturesque imagery set forth in ordinary language. But Henri Poincaré often happily rejuvenated the common expression by carrying to a conclusion the comparison that it implies or by imbuing a figure employed with originality, freshness, appealing power. . . .

"In scientific matters his only preoccupation was the search for truth. He concerned himself little with glory. He preferred that his name should not be given to any of his discoveries: to contemplate truth for an instant face to face was the only reward which appeared to him worthy of emulation. In acting thus he was certainly obedient to considerations of a lofty order, of an æsthetic order if one may so express it; but he was swayed by an impetuous sense of justice. To him the savants were all soldiers of one army. If in the common contest to ravish nature of her secrets some brilliant captains organize victory, it is nevertheless owing to the discipline, to the courage, to the endurance of the troops, that they win so completely. But, in the combat, how many brave soldiers fall 'sans laisser de noms, et après avoir utilement aidé à la victoire'"! . . .

"He had the happiness to unite his life to that of an intelligent companion, discreet and devoted, who embellished his existence and facilitated his tasks; for as M. Appell has expressed it, 'elle entourait son mari de l'atmosphère familiale, profondément unie et calme, qui seule permet les grands travaux de la pensée.' Henri Poincaré was the most tender and happy of fathers. He saw growing about him the children whom he loved profoundly and who recompensed him by their care always to have lurking about their lips the sweet smile of their affection and their joy. . . .

"As supreme consolation to those who loved him, Henri Poincaré has bequeathed to the centuries to come, with the example of a life as simple as it was beautiful and nobly completed, sa réconfortante pensée, sa foi en la grandeur, en la beauté de l'humanité. Son exemple et son oeuvre ont vaincu le néant.

"'Recevant d'âge en âge une nouvelle vie,
Ainsi s'en vout à Dieu les gloires d'autrefois;
Ainsi le vaste écho de la voix du génie
Devient du genre humain l'universelle voix. . . .'"

Although the new edition of Lebon's bio-bibliographical work is about a half larger than the old one* its general plan and appearance is the same. To the section on Biography have been added: (1) Darboux's reply to Poincaré's discourse at Darboux's Jubilee; (2) corrections and additions to the list of degrees, honorary titles, decorations, etc.; (3) titles of articles and works on Poincaré. The bibliographical part of Section II, on Mathematical Analysis, has been increased in size by 9 pages through the addition of details concerning new editions, reviews and new titles. In a similar way 5 pages have been added to Section III, on Analytical Mechanics and Celestial Mechanics, 6 pages to Section IV, on Mathematical Physics, and 2 pages to Section V, on Scientific Philosophy. History of Sciences is the title of Section VI, instead of "Necrology" in the old edition; the bibliography has been increased fourfold. Section VII is much the same. The total number of titles is 495, an increase of 59.

This work has been most admirably carried out and is beautifully printed and arranged. Errors and omissions at the date of publication, May 25, 1912, are probably very few. The error (page 85) in the page numbers (583 for 593) of Poincaré's note on non-euclidean geometry in the Traité de géométrie of Rouché and Camberousse† still persists. And Russian editions‡ of the conférence on "L'évolution des lois" (page 92) are overlooked.

As practically no reference has been made in this BULLETIN to recent memoirs on Poincaré, I append herewith a list of titles which I have met with, and which supplement Lebon's work. Those memoirs which are signed, are arranged alphabetically according to authors. It will be generally conceded, I believe, that Hadamard's memoir, on Poincaré as a mathematician, is of unsurpassed excellence.

R. d'Adhémar, (1) *Revue des questions scientifiques*, Brussels, vol. 22 (3), 1912, pp. 349–385; (2) Henri Poincaré ("Philosophes et Penseurs" série), Paris, Bloud et Gay, 1914. 12 mo. 64 pp.

---

* Reviewed by J. W. Young in this BULLETIN, October, 1910, vol. 17 (2), pp. 42–43.

† There are the same page numbers (581–593) for this note in the 8e éd. of the Traité, Paris, 1912.

‡ *Kagan's Bote* (formerly *Spaczinski's Bote*), Nr. 544, pp. 81–89 and Nr. 545, pp. 105–112, 1911. Also separately printed with E. Kohn's "Space and time from the standpoint of physics" under the title, in Russian, "Kohn and Poincaré: Space and time from the standpoint of physics." Odessa, *Mathesis*, 16mo., 81 pp.

—P. Appell, (1) *Revue du mois*, vol. 14, 1912, pp. 129–132; (2) *Annuaire Bureau des longitudes*, 1913, D. 14–18; (3) *Revue scientifique*, vol. 19, 1913, pp. 475–476, vol. 20, 1913, pp. 144–146.—K. Bajev, *Nouvelles de la société russe d'astronomie* (Russian), vol. 7, 1912, pp. 263–269.—R. Berthelot, Un romantisme utilitaire; étude sur le mouvement pragmatiste. Tome 1: Le pragmatisme chez Nietzsche et chez Poincaré. Paris, Alcan, 1911. (Deuxième partie; un pragmatisme scientifique. le pragmatisme fragmentaire et mitigé de Poincaré, pp. 195–416.)—Bigourdan, *Annuaire Bureau des longitudes*, 1913, D. 19–23. —E. Boutroux, *Revue de Paris*, vol. 20_1, pp. 673–702, vol. 20_2, pp. 77–91, 1913; also reprinted in pamphlet form, Coulomniers, 1913, 47 pp.—P. Boutroux, *Revue du mois*, vol. 15, 1913, pp. 155–183.—H. C. Brown, *Journal of Philosophy, Psychology and Scientific Methods*, New York, vol. 11, 1914, pp. 225–236.—L. Brunschvicg, "Poincaré le philosophe," *Revue de métaphysique et de morale*, vol. 21, 1913, pp. 585–616 (portrait of Poincaré in 1887.)—A. Buhl, *L'Enseignement Mathématique*, vol. 15, 1913, pp. 9–32 (portrait of Poincaré in his study).—R. H. Chassériaud, *Elektrotechnische Zeitschrift*, Berlin, vol. 33, 1912, p. 883.—J. Claretie, *Annuaire Bureau des longitudes*, 1913, D. 3–7.—Cornille, *Annuaire Bureau des longitudes*, 1913, D. 23–25.—S. Dickstein, *Wiadomości matematycne*, Warsaw, vol. 16, 1912, pp. 249–260 (portrait).—J. Echegazay, *Revista Soc. matem. española*, Madrid, vol. 2, 1912, pp. 33–39 (portrait).—G. Eichhorn, *Jahrbuch der drahtlosen Telegraphie*, Leipzig, 1912, vol. 6, pp. 109–113.—H. Fehr, *L'Enseignement Mathématique*, vol. 14, 1912, pp. 391–392.—M. Fouché, "La philosophie d'Henri Poincaré," *Bulletin de la Société Astronomique de France*, Paris, vol. 27, 1913, pp. 299–306. (Conférence faite à la séance du 2 Octobre, 1912.)—Galazine, *Bulletin de l'Acad. Impériale des Sciences de St. Pétersbourg* (Russian), 1912, pp. 819–820.—Gust'Hau, *Annuaire Bureau des longitudes*, 1913, D. 1–3.—J. Hadamard, (1) "Poincaré le mathématicien," *Revue de métaphysique et de morale*, vol. 21, 1913, pp. 617–658 (portrait of Poincaré in 1908); (2) "Henri Poincaré et le problème des trois corps," *Revue du mois*, vol. 16, 1913, pp. 385–418.—S. C. Haret, *Bulletin* (section scientifique) *Acad. roumaine*, Bucharest, vol. 1, 1912–13, pp. 50–65.—G. Humbert, *La Nature*, vol. 40_2, 1912, pp. 143–144 (1887 portrait, photo by Pirou).—P. E. B. Jourdain, *The Monist*, vol. 22, 1912, pp. 611–615.—A. Korn, *Sitzungsberichte d. mathem. Ges.*, Berlin, vol. 12, 1913, pp.

2–13 (portrait).—E. Lampe, *Naturwissenschaftliche Rundschau*, Braunschweig, vol. 27, 1912, pp. 476–479.—P. Langevin, " Poincaré le physicien," *Revue de métaphysique et de morale*, vol. 21, 1913, pp. 675–718; the same in *Revue du mois*, vol. 16, 1913, pp. 419–463.—A. Lebeuf, " Poincaré l'astronome," *Revue de métaphysique et de morale*, vol. 21, 1913, pp. 659–674.—Lippmann, (1) *Annuaire Bureau des longitudes*, 1913, D. 7–9; (2) *Comptes rendus de l'Acad. d. Sciences*, Paris, vol. 155, 1912, pp. 1280–1283.—A. E. H. Love, *Proceedings of the London Math. Society*, vol. 11 (2), 1913, pp. xli–xlviii.—P. Mansion, *Mathesis*, vol. 2 (4), 1912, pp. 233–238.—L. Margaillan, *Internationale Monatsschrift für Wissenschaft*, vol. 7, 1912, pp. 546–555.—F. Masson, *Revue scientifique*, vol. 18 (5), 1912, pp. 628–629.—C. Meyer, *Anales Soc. scientif. Argentina*, Buenos Ayres, vol. 74, 1912, pp. 125–147.—A. Mieli, *Rivista di filosofia*, vol. 5, 1913, pp. 44–48.—G. A. Miller, *Science*, New York, vol. 36 (2), 1912, pp. 425–429.—K. Mittenzwey, *Nord und Süd*, Berlin, vol. 147, 1913, pp. 53–58.—Morduchaj-Boltovskoj, *Bulletin de l'Université Impériale de Varsovie* (Russian), vol. 24, 1913, pp. 27–80.—L. T. More, " Poincaré and the Philosophy of Science," *The Nation*, New York, vol. 95, 1912, pp. 242–244.—F. R. Moulton, *Popular Astronomy*, vol. 20, 1912, pp. 621–634.—C. Nordmann, *Revue des deux mondes*, vol. 445, 1912, pp. 331–368.—L. Octavio de Toledo, *Revista soc. matem. espagñola*, Madrid, vol. 2, 1912, pp. 26–27.—P. Painlevé, (1) *Revue du mois*, 1912, pp. 132–134; (2) *Annuaire Bureau des longitudes*, 1913, D. 9–13.—E. Pascal, (1) *Giornali di matem.*, vol. 3 (3), 1912, pp. 303–309; (2) *Rendiconto Accad. d. Sc.*, Naples, vol. 18 (3), 1912, pp. 309–313.—E. Picard, *Annales scientif. de l'Ecole normale sup.*, Paris, vol. 30 (3), 1913, pp. 463–482; also printed separately.—G. Rageot, Les savants et la philosophie. Paris, Alcan, 1908. (Chapter 2: Le néo-criticisme d'un géomètre Henri Poincaré.)—L. Rougier, Henri Poincaré et la mort des vérités. Paris, *La phalange*, 1913, 22 pp.—G. Sarton, (1) *Ciel et terre*, Brussels, 1913, 25 pp. (portrait); (2) *Isis*, vol. 1, 1913, pp. 95–97 (portrait).—J. B. Shaw, " Poincaré as an Investigator," *Popular Science Monthly*, New York, vol. 82, 1913, pp. 209–224.—E. E. Slosson, "Major Prophets of to-day," *Independent*, vol. 71, 1911, pp. 729–741 (portrait); also in volume: Major Prophets of to-day, Boston, 1914, pp. 104–146.—W. B. Smith, *The Monist*, vol. 22, 1912, pp. 615–617.—C. Somigliana, *Atti della Reale Accademia della Scienze di*

Torino vol. 49, 1914, pp. 45–54.—J. W. N. Sullivan, *Scientific American*, vol. 107, 1912, p. 78 (portrait).—I. Tschistiakov, *L'Enseignement Mathématique* (Russian), vol. 5, 1912, pp. 197–199.—G. Tzitzéica, *Gazeta matematica*, Bucharest, vol. 17, 1912, pp. 441–445.—O. Veblen, *Proceedings of the American Philosophical Society*, vol. 51, 1912, 9 pp.—V. Volterra, "Henri Poincaré: l'Oeuvre mathématique," *Revue du mois*, vol. 15, 1913, pp. 129–154.*—A. G. Webster, "Poincaré as a mathematical physicist," *Science*, vol. 38 (2), 1913, pp. 901–908.—H. Weyl, *Mathematisch-naturwissenschaftliche Blätter*, vol. 9, 1912, pp. 161–163.†

A few anonymous notes and sketches are to be found in: *Bulletin of the American Mathematical Society*, vol. 19 (2), 1912, p. 43.—*Nature*, vol. 90, 1912, pp. 353–356 (portrait plate supplement and autograph).—*Revue générale des Sciences*, vol. 23, 1912, p. 533.—*Revue scientifique*, vol. 17, 1912, p. 90.—*Sammlung und Mitteilungen und Protokolle der Math. Gesellschaft in Charkow* (Russian), vol. 13 (2), pp. 4–5.—*Supplemento ai Rendiconti del circolo matematico di Palermo*, vol. 8, 1913, pp. 13–32 (reprint of extracts from Lebon's work together with a facsimile of a Poincaré letter).—*Times*, London, July 18, 1912, p. 9, col. c.

In addition to the portraits listed above there is a full-page portrait and autograph in the *American Journal of Mathematics*, vol. 12, 1890; this is the 1887 portrait reproduced in the *Revue de métaphysique et de morale* and *La Nature*. An early portrait is reproduced in *Acta Mathematica*, 1882–1912, *Tables générale des Tomes 1–35*, Upsala et Stockholm, 1913, p. 164. In *Popular Science Monthly*, vol. 82, 1913, p. 412, there is a poor reproduction of the admirably life-like heliogravure frontispiece to Lebon's bio-bibliographical work.

Of the portraits above noted the best are those in the *American Journal etc.*, *L'Enseignement Mathématique*, *Revue de métaphysique etc.*, *Nature*, and Lebon's work.

---

*The four memoirs by P. Boutroux, Hadamard, Langevin and Volterra, published in *La revue du mois*, have also been issued in book form with five pages of supplementary matter. Paris, Alcan, 1914. 12mo. 2 + 265 pp.

† It may be of interest to add a reference to a sketch of Poincaré prepared by Goursat. This consists mainly of extracts from the writings, mentioned above, of Lebon, Masson, P. Boutroux, V. Volterra, Humbert and Painlevé. The sketch is to be found in La vie et les travaux des savants modernes par A. Rebière; troisième édition revue et augmentée par E. Goursat. Paris [1913], pp. 188–203.

It is to be hoped that in the next edition of this work M. Lebon may be moved to give a list of papers and books which have been inspired by Henri Poincaré's suggestions and discoveries.

R. C. ARCHIBALD.

BROWN UNIVERSITY,
PROVIDENCE, R. I.

SHORTER NOTICES.

First-Year Mathematics for Secondary Schools. By ERNST R. BRESLICH. Chicago, The University of Chicago Press. 1915. 344 pp.

ABOUT six hundred fifty years ago Roger Bacon gave voice to his feelings with respect to the teaching of mathematics, and this voice was in no respect uncertain nor was it at all lacking in emphasis. His words may be found in the Opus Majus, in the Opus Tertium, and in the manuscripts as yet unpublished of his De Communia Mathematicæ. In the last-mentioned work Bacon says that students are burdened with unnecessary difficulties to such a degree that they come to despise mathematics, whereas, if properly taught, the subject could be understood without any unreasonable expenditure of time; and that the first course in mathematics should not be designated as geometry, arithmetic, and so on, but as the elements of mathematics, a preliminary to the special branches.

What Bacon had to say on this phase of teaching was not new; others had said it before, and thousands have said it since, and after a fashion many have put the idea into practice. And so the effort of Mr. Breslich comes to the teaching profession as merely an ancient one clad in new guise. This does not in the least detract from the laudable nature of the effort, but it serves to give the work a kind of historical setting which assists us in judging of its novelty and its probable effect upon education.

The central idea of the work seems to be to select those features of secondary mathematics which are easily within the reach of beginners, postponing the consideration of the more difficult ones to a later period. As the author puts it, "The simpler principles are best suited for beginners, and may therefore be brought together in an introductory course."

In the pursuit of this idea the author proposes to treat of algebra and geometry at the same time, thus carrying out the ancient idea of fusion to which reference has been made above. He also proposes to consider those "subjects in which practical values are most clearly exhibited," to introduce a certain amount of trigonometry, to avoid "formalism in mode of presentation," and to give the student a "broader mathematical preparation."

With most of this ideal the educational world is generally in sympathy—perhaps with all of it except the mixing of algebra and geometry with no definite system. The following questions, however, will naturally arise in the minds of all who have to consider the book: Has the author carried out the plan successfully? That is, does the book meet the ideals which he has himself laid down? Given the average teacher, will the student, at the end of his work in the high school, be as well grounded in mathematics as he would have been if the work had been arranged on some other plan? Will he appreciate the subject as well or be as apt to continue his study of its higher branches?

In answer to the first question the reader is likely to hesitate before committing himself to an affirmative. He will find fully as much formalism in the early pages (for example, pages 5, 12, 20, 23) as he will find in any of the older types of algebra or geometry; he will find the commutative and associative laws given much earlier than the experience of teachers generally sanctions; he will find, for a work of this nature, an excessive number of definitions; he will find the rules of operation as formally stated as in the more common type of textbook; he will find the euclidean form of greatest common measure, with applications to numbers as large as those in the text-books of two generations ago; he will find such problems as the division of \$2,400 into two parts having the ratio of 2 : 1 quite as he would find them in other books; he will find the simple made difficult in various cases, as in such products as $(a - b)c$, $(a + b)(a - b)$, $(a - b)(c + d)$, and $(a - b)^2$, and in the law of signs in multiplication as based on the "turning-tendency" idea; and he will find much the same type of problem that has come down to us from the past, as about a field that is twice as long as wide, and if it were 20 rd. longer and 24 rd. wider the area would be doubled. And when the serious inquirer has finished his reading of the book

he will have a feeling of doubt as to whether the plan of improving the first steps in mathematics has been as successful as he had hoped it would be.

And similarly as to the second question. Of course the book can be successfully taught; that is true of any book, provided the right teacher is available. But that a book with what seems to be a forced fusion of essentially different branches of a science, based solely upon the theory of ease of presentation, which theory does not seem to have been carried out—that such a book can be generally successful can hardly be expected.

It seems unfortunate that there should be in the book such statements as that "Pacioli in 1494 was the first to give rules for all processes of addition, subtraction, multiplication, and division" (page 20); that Tartaglia should be spoken of commonly as Fontana (facing page 158), when he himself preferred the former name, as the titles of his books prove; that Vieta or Viète should appear as Vièta (page 210); that the title of Fibonacci's work should be given as *Algebra et almuchabala*, when the manuscript actually begins "Incipit liber Abaci a leonardo filio Bonacij Pisano"; that the student should meet with the name Alkarismi (facing page 213) and with the inexcusable transliteration (unless with diacritical marks) of Al Hovarezmi a few pages later (page 255); that he should be told (page 20) that Diophantus lived about 250, and later (page 281) that he lived in the fourth century; and that numerous other slips of this kind should have been made in preparing the text.

The statement that "the coefficient of any factor in a term is the product of all the other factors of the term" will not seem very clear to a student who is told that 2 is the coefficient of x in the term $2x$. The treatment of negative numbers in Chapter XII will probably not seem to most teachers as clear as those to be found in our common algebras. The assertion that $a \times 0 = 0 \times a$ (page 195) will not seem warranted to those who may use the book, since the commutative law has not been referred to with respect to zero. Such problems as Example 31 on page 251, Example 10 on page 257, and numerous others of this type, will not lead teachers to feel that the author has broken away from the poorest type of inherited puzzles. These and criticisms like these will doubtless strike even the casual reader, and the causes for them will be regretted by all who wish success to any venture of this nature.

In the matter of skillful mathematical typography the book leaves more to be desired than is usually the case.

DAVID EUGENE SMITH.

Les Coordonnées intrinsèques, Théorie et Applications. Par L. BRAUDE. (Scientia, série physico-mathématique, no. 34.) Paris, Gauthier-Villars, 1914. 100 pp. Price 2 francs.

IN 1849 and 1850 William Whewell read two memoirs* on the intrinsic equation of a curve and its applications, before the Cambridge Philosophical Society. The opening paragraph of the first memoir is as follows:

"Mathematicians are aware how complex and intractable are generally the expressions for the lengths of curves referred to rectilinear coordinates, and also the determinations of their involutes and evolutes. It appears a natural reflexion to make, that this complexity arises in a considerable degree from the introduction into the investigation of the reference to the rectilinear coordinates (which are *extrinsic* lines); the properties of the curve lines with relation to these straight lines are something entirely extraneous, and additional with respect to the properties of the curves themselves, their involutes and evolutes; and the algebraical representation of the former class of properties may be very intricate and cumbrous, while there may exist some very simple and manageable expression of the properties of the curves when freed from these extraneous appendages. These considerations have led me to consider what would be the result if curves were expressed by means of a relation between two simple and *intrinsic* elements; the length of the curve and the angle through which it bends: and as this mode of expressing a curve much simplifies the solution of several problems, I shall state some of its consequences." He then considers the curve defined by the equation

$$s = f(\varphi), \tag{1}$$

points out that the radius of curvature follows at once from the relation

$$\rho = \frac{ds}{d\varphi} = F(\varphi), \tag{2}$$

* *Transactions of the Cambridge Philosophical Society*, vol. 8, part 5 (1849), pp. 659, 671; vol. 9, part 1 (1850), pp. 150, 156.

whence the equation of the evolutes,

$$s' = \frac{ds}{d\varphi} + C.$$

Similar relations result for successive evolutes and involutes.

Whewell then applies his discussion to the circle, equiangular spiral, cycloid, epicycloid, hypocycloid, "running pattern curves," catenary, and tractrix. He also derives some general properties, such as the derivation of the intrinsic equation of a curve when given its equation in rectangular coordinates and vice-versa, with applications to the parabola, ellipse, and semi-cubical parabola.

The second memoir contains further applications.

In connection with his first memoir Whewell remarks: "After writing this paper I found that Euler had, in the solution of a particular problem, expressed curves by means of an equation between the arc and the radius of curvature. This equation is, as is shown in the paper, the differential of my "intrinsic equation," and has an equally good right to the name. My equation being the integral of Euler's has, of course, one more arbitrary constant than his. There may very possibly be other modes of expressing curves which may be fitly described as "intrinsic equations" to the curves. I was not able to find any other name for the equation which I employed." No definite reference is given to Euler's works; but complete data of this nature, in connection with eight different papers, the first published in 1738 and the last in 1824, are given in Wölffing's bibliography.*

It may be remarked that while Whewell's memoirs† undoubtedly served to inspire the later elaborate developments of intrinsic geometry, and while French and Italians adopted Whewell's name "intrinsic," the same system of coordinates was also used, without special indication of its applications, by the German philosopher K. C. F. Krause (1802, 1804,

* E. Wölffing, "Bericht über den gegenwärtigen Stand der Lehre von den natürlichen Koordinaten" [in two dimensions]. *Bibliotheca Mathematica*, vol. 1(3) (1900), pp. 142–159. References are given to papers and books of nearly 90 different authors. M. Braude gives (p. 12) an incorrect reference to *Bibliotheca Mathematica*, vol. 2(3) (1901).

Intrinsic coordinates are discussed in the Encyklopädie der Math. Wissenschaften, Bd. III, 379 ff.; and Bd. III–3, pp. 34 ff., 84 ff., 198, etc.

† These memoirs were translated into German by A. Walter, Jahresbericht 1907 der k. k. ersten Staatsoberrealschule, Graz.

1835) and by A. Peters (Neue Curvenlehre, 1838), who is said to have introduced (Braude, page 9) the German term "natürliche Koordinaten" as opposed to Cartesian coordinates.

If equation (2) be looked upon as the polar equation of a curve, this curve is said to be the "radial curve" or radial of the original curve. These curves were so named by Robert Tucker who commenced their study in 1863. Four of his five papers on the subject are listed by Wölffing.* Tucker's definition was: "If from a point straight lines are drawn equal and parallel to the radii of curvature at successive points of a curve, their extremities will trace out the Radial Curve corresponding to the given curve."

The usual form of the intrinsic equation now employed is

$$f(s, \rho) = 0, \tag{3}$$

where ρ is the radius of curvature at a point of the curve defined by a given s. In a paper published in 1741 Euler discussed when an equation of this form defines an algebraic curve. If s and ρ be regarded as the rectangular coordinates of a point we get what Wölffing has called (1899) the Mannheim curve of the primitive curve, since it was in a memoir of 1859 that Mannheim remarked, among other properties, that the locus of the centers of curvature of the points of contact of this curve as it is rolled along a straight line is the curve defined by the intrinsic equation (3).

Again, if we consider the envelope of the line

$$x + y \tan \varphi - s = 0,$$

where s is the portion of the x-axis, measured from the origin, which is cut off by the given line, and φ the complement of the angle which the line makes with the x-axis, we get the tangential equation of the curve in the form of equation (1). Such a tangential equation has been studied at length by Casey and others. In a recent thesis by Koestlin,† however, this curve has been considered in its relation to the curve whose intrinsic equation is (1) and he has called it the "arcuïde" of the curve (1).

* The paper which Wölffing missed is in Mathematical Questions with their Solutions from the *Educational Times*, vol. 4, 1866, pp. 22–28.

† Koestlin, Ueber eine Deutung der Gleichung, die zwischen dem Bogen einer Kurve und dem Neigungswinkel der Tangente im Endpunkte des Bogens einer Kurve besteht. Tübingen, 1907.

The semi-intrinsic equations

$$f(s, x) = 0, \quad f(s, y) = 0$$

between the length of arc of a curve and the abscissa or the ordinate of the variable, have been discussed by several writers.

Sylvester and others have studied the analogous equation

$$f(s, r) = 0$$

between the arc and the radius vector.

Such are some of the principal forms of intrinsic equations used in the discussion of plane curves and their relations with one another.

The special importance of intrinsic coordinates was first brought out by Sophus Lie in his discussion of certain differential invariants in the theory of groups.* But this subject, as well as many applications of the coordinates to problems in differential and other geometry of curves and surfaces, is not considered in Braude's booklet. For these the reader must turn for guidance to the Encyklopädie or to the elaborate work of Cesàro: Lezioni di Geometria Intrinseca, first published at Naples in 1896.†

In the first 43 pages Braude gives "développements et méthodes" with numerous illustrative examples. Some subjects treated are: asymptotes, envelopes, successive evolutes, osculating curves, contact of higher orders, systems of curves and invariants of a system, and parallel curves. Special curves such as the epicycloids and hypocycloids, conics, logarithmic spiral, and causticoide are introduced in the examples.

The second section (pages 43–68) of the little book is taken up with a discussion of La Courbe de Mannheim. It is first introduced, as above, with a straight line as base; the generalization to a circle as base by Wieleitner and others is also developed with examples. And finally the circle is replaced as base by any plane curve. Another form of generalization

* Lie-Scheffers, Differentialgleichungen mit bekannten infinitesimalen Transformationen, Leipzig, 1891.

† M. Braude gives this date incorrectly (p. 6) as 1895. A German edition by G. Kowalewski: Vorlesungen über natürliche Geometrie, was published at Leipzig in 1901. Both editions were reviewed in this BULLETIN, vol. 9, Apr. 1903, pp. 349–357, by V. Snyder. The Italian edition was also reviewed by E. O. Lovitt, vol. 5 (March, 1899), pp. 303–306.

of Mannheim's curve was treated by that geometer himself and consists in seeking "the locus $M^{(n)}$ of the centers of curvature of order n"; $M^{(2)}$ would be the locus of the centers of curvature of order 2, that is, the evolute of $M^{(1)}$, the (first) Mannheim curve of the rolling curve. Then the general Mannheim curve $M^{(n)}$ (C, Γ) of a curve C, and with any plane curve Γ as base, is also discussed. Paragraphs on "intermediate evolutes," radials and the comparison of singularities, of corresponding radials and the curves of Mannheim, conclude the section. Again in the numerous examples many special curves are introduced. For example: (1) with a straight line as base, the Mannheim curve of a catenary is a parabola; (2) when the pseudo-cycloid or the logarithmic spiral is rolled on a curve Γ, we get as $M^{(4n)}$ (C, Γ) the curve Γ itself; $M^{(4n+3)}$ (C, Γ) is an involute of Γ.

The third section (pages 68–80) deals with the arcuïde. In particular there are sections on arcuïdes of algebraic curves, generation of the arcuïdes as glissettes, and on the generalization of the arcuïde by replacing the straight line by a curve as base (as in the case of Mannheim's curve).

The last section is entitled Les Roulettes, but without taking up too much space it is scarce possible to indicate further the nature of the discussion. Several theorems of Besant's Notes on roulettes and glissettes are recalled. M. Braude would have found still other interesting examples in the notable memoir written by the physicist J. Clark Maxwell at the age of seventeen: "On the theory of rolling curves."*

The work is highly accurate and takes account of recent developments of the subject, many of which are due to the author himself. The figures are admirably clear, although their face is sometimes hardly in keeping with the letter press. The style is concise but the numerous illustrative examples simplify the presentation of the theory which, in the very nature of the case, gives one the impression of being somewhat scrappy.

The volume is a very useful addition to the Scientia series and constitutes a pleasant introduction to Cesàro's great work.

R. C. ARCHIBALD.

---

* *Proc. Royal Soc. Edinb.*, vol. 16, Feb., 1849, pp. 519–540; Collected Works, edited by Niven, 1890, vol. 1, pp. 4–29.

Leçons sur la Théorie des Nombres. Par A. CHÂTELET. Paris, Gauthier-Villars, 1913. x+156 pp.

THIS little volume consists, as the preface tells us, of the Peccot Foundation lectures in form substantially as delivered at the Collège de France in the second semester of the year 1911–12. Following the line of development given in his thesis, "Sur certains ensembles de tableaux et leur application à la théorie des nombres" (*Annales scientifiques de l'Ecole Normale Supérieure*, 1911), the author bases his exposition of the Dedekind theory of moduli and ideals largely upon the geometrical ideas of Minkowski and the so-called method of continued reduction of Hermite.

In the first chapter the algebraic foundation is laid by giving some necessary theorems concerning matrices and their relation to sets of forms, together with an all too brief account of Minkowski's theory of generalized distance. As in the thesis the term "*tableau*" is used throughout for a square matrix and the term matrix is employed to denote a rectangular array.

In the four following chapters the theory of Dedekind's moduli with its applications is studied in detail. If the sum and difference of two points be defined as in vector addition so that

$$(p_1, p_2, \cdots, p_n) \pm (q_1, q_2, \cdots, q_n) = (p_1 \pm q_1, p_2 \pm q_2, \cdots, p_n \pm q_n),$$

the definition of a modulus of points in agreement with Dedekind's definition of a modulus of numbers follows naturally. It is then easy to show that the coordinates of the points of the simplest modulus of dimension m in a space of dimension n are given by the matrix equation

$$||p_1, p_2, \cdots, p_n|| = ||x_1, x_2, \cdots, x_n|| \times A,$$

where the x's are integers and A is a matrix with m rows and n columns. The modulus is said to be "type" if A exists with the elements of each row coordinates of a point of the modulus such that every point of the modulus is given by the matrix equation. The matrix A is called a "base" of the modulus. The criterion for a type modulus is that it has only a finite number of points all of whose coordinates are less in absolute value than a given number. In general there are only a finite

number of points whose distance from a given point of the modulus is finite. The points of the modulus form a lattice work.

Passing to realms of algebraic numbers, the author interprets the n conjugate values of a number $\bar{\omega}$ in a realm $K(\omega)$ as the coordinates of a point in space, real or semi-real, of n dimensions. The elementary theorem which affirms that $\bar{\omega}$ may be expressed as a linear integral function with rational coefficients of the n powers of a primitive element ω of $K(\omega)$, gives the matrix equation

$$||\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n|| = ||a_0, a_1, \cdots, a_{n-1}|| \times \Omega,$$

where the a's are rational and Ω is the square root of the discriminant of ω. The numbers of the realm therefore give rise to a modulus R within which the integers of the realm form a type modulus C. The sub-modulus C is then according to Dedekind a multiple of R.

In the matrix equation for the conjugate values of an algebraic integer, the table Ω may be replaced by any other table

$$\Pi = R\Omega,$$

where the elements of R are rational and its determinant is not zero. It is then possible to determine π to a factor près in such a way as to establish an isomorphism between the numbers of the realm and the abelian system of tables

$$X_{\bar{\omega}} = \pi[\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n]\pi^{-1}$$

with *rational* elements, where the bracket denotes a canonical table, i. e., a table in which the numbers $\bar{\omega}_i$ occupy the principal diagonal and all other elements are zero.

Similarly, when the integers of a realm are studied, it is easily shown that the integers of any ideal of the realm form a sub-modulus A of C which is again type and which is given by the matrix equation

$$||\beta_1, \beta_2, \cdots, \beta_n|| = ||x_1, x_2, \cdots, x_n|| \times PT,$$

where the x's are integers, T is a base of the realm, and P is a table with rational elements. The table PT will be " a base relative to an ideal " if, and only if, the elements of the table

$$(PT)[\alpha_1, \alpha_2, \cdots, \alpha_n](PT)^{-1}$$

are integers for every integer α of the realm.

In the two remaining chapters, devoted to continual reduction and the theorems of Minkowski and to the reduction of a base of a realm, geometrical ideas are brought even more clearly into evidence. In a modulus of points of dimension n in space of n dimensions, the points give rise to a totality of tables each of which is formed of the coordinates of n linearly independent points. Among the tables of such a totality it is natural that we should seek for the simplest. But what shall be the criterion? Geometrical intuition suggests at once that we choose those tables whose points lie as near as possible to the origin. But here a difficulty confronts us, for it may be that several points are equally distant from the origin and we cannot choose between them. These difficulties are overcome if for the ordinary distance we substitute Minkowski's "span," which is defined as the maximum of the absolute values of the differences between the coordinates of the two points. Thus the span $S(OA)$ of a point A from the origin is the maximum of the absolute values of its coordinates. Further, when two points have the same span with respect to the origin, we may distinguish between the ranks of their spans if we define "rank of span" to be the number reading from left to right of the maximum coordinate. For example, the two points A_1 (1, 5, 5) and A_2 (2, 3, 5) have the same span but the rank of A_1 is 2 while the rank of A_2 is 3.

A simplest or "reduced table" is one whose rows are the coordinates of n linearly independent points $A_i(p_1^{(i)}, p_2^{(i)}, \cdots, p_n^{(i)})$ which satisfy the following conditions:

(1) $$S(OA_1) \leqq S(OA_2) \leqq \cdots \leqq S(OA_n),$$

(2) $$p_1^i > 0,$$

(3) If $S(OA_i) = S(OA_{i+1})$, rank $S(OA_i) \leqq S(OA_{i+1})$.

For a given modulus the number of reduced tables is finite, as is geometrically evident. Nevertheless, it is possible to subject the modulus to a dilatation with continuous parameters so that for each of an infinite number of systems of values of the parameters a finite number of reduced tables exist. The importance of this method of introducing continuous parameters, which is Hermite's method of continuous reduction, lies in the fact that in a very important case, namely the case where the table is real and consequently associated with a decomposable form, the table is given only to a dilatation près.

When $n = 2$ and the elements of the table are real the series of reduced tables may be arranged, one for each interval, as a single parameter λ ranges both ways from a given value λ_1 defining a series of intervals such that

$$0 < \cdots < \lambda_{-2} < \lambda_{-1} < \lambda_1 < \lambda_2 < \cdots < \infty, \qquad (\lim \lambda_i = \infty, \lim \lambda_{-i} = 0).$$

For the general case the reduction is accomplished by means of the fundamental inequalities of Minkowski. The two fundamental problems of finding the units of a realm and of separating the ideals of a realm into classes of equivalent ideals are made to depend upon the reduction of a base.

Three notes, the first on the application of the theory of moduli to periods of functions, the second a study of the realm $K(\sqrt{82})$, and the third a brief account of congruences with respect to an ideal and with respect to the norm of an ideal, occupy the last twenty-four pages of the book.

M. Châtelet modestly disclaims for the book any originality so far as material is concerned. But if it contains no hitherto unpublished results, the treatment is sufficiently novel to make the book a noteworthy contribution to the literature of algebraic numbers. The bringing together of the algebraic analysis of Hermite and the geometrical researches of Minkowski as aids to the development of the brilliant conceptions of Kummer and Dedekind is an achievement for which the mathematical world owes much to the author.

The book as a whole is well written, though at times it is brief almost to the point of obscurity. For the ordinary reader its value would have been greatly enhanced by additional concrete illustrations, and by even a few figures similar to those which illuminate Minkowski's Diophantische Approximationen.

E. B. Skinner.

A Treatise on the Analytic Geometry of Three Dimensions. By George Salmon. Fifth edition, volume 2, edited by R. A. P. Rogers. London, Longmans, Green and Company, 1915. xvi+334 pp.

The second volume of the fifth edition of the Treatise begins with families of surfaces, which was Chapter XIII of the fourth edition. The numbering of the chapters has been retained.

The first section has not been changed; the second, on line geometry, has been enlarged from six to forty pages, and made into a new chapter, XIII*a*. It first considers the linear complex, derives the canonical equation, and gives a few important examples. After defining a quadratic complex, the treatment concerns differential properties, including a discussion of singular complexes, singular surface, etc., and then applies the results to the quadratic complex. The singular surface is obtained and a number of properties derived, including the existence of the six special congruences. Then follow a few general properties of congruences of lines, focal surfaces, limiting surface, developables, normal and isotropic congruences. Too many interesting properties are crowded into fine print and given as examples. The part on ruled surfaces is hardly changed.

Section III of Chapter XIII of the fourth edition is also made into a new chapter, XIII*b*. The change consists of an addition of fourteen pages on Lamé's curvilinear coordinates, and Darboux's theorem concerning the intersections of orthogonal surfaces; normal congruences of curves and cyclic systems close the chapter.

In Chapter XIV some bibliographical references are inserted at the beginning. Only one short article on inversion is added. The chapter on the cubic surface is amplified by various minor additions and numerous examples, but the general treatment is not changed.

A number of additions to the theory of the quartic surface have been made; much modern literature has been cited, but by no means all of the important memoirs have been given.

The text of the former edition has been so faithfully reproduced that a serious error concerning plane sections of a ruled surface, passing through a straight line director, is found in the new book (page 203). Two pages are given to Steiner's quartic, one to mapping a quartic surface with a double line on a plane. A short explanation of the complex surface of Plücker, and a fuller discussion of the cones of Kummer connected with the general cyclide, a few pages on the Weddle surface giving an incomplete summary of the later development of this theory, an amplified description of the symmetroid, and the explanation of the Kummer surface rewritten constitute the changes in this chapter.

In the section on the transformations of surfaces, an intro-

duction to the general theory of Cremona transformations is inserted, but so brief and containing so many statements without proof that its value is much less than could justly be expected. A fairly full discussion of the (2, 2) case is given, also an outline of the (2, 3) and (2, 4) cases, but no claim is made for completeness. A brief statement concerning Segre's work on the resolution of singularities by means of quadratic transformations is misleading. The problem was by no means completely solved by Segre.

With the tremendous amount of new material that has been contributed to this field during the last few decades it is difficult to develop a systematic theory without expanding into several volumes. The editor has endeavored to present these new phases in as near the same degree of completeness as was adopted by the author toward the corresponding field when the last preceding edition was written. This has, on the whole, been accomplished. The same style has been followed and a book has been produced, which "forms, it is hoped, a concise and comprehensive survey of tri-dimensional euclidean geometry, both algebraic and differential."

VIRGIL SNYDER.

Hermann Grassmanns gesammelte mathematische und physikalische Werke. Herausgegeben von FRIEDRICH ENGEL. Bd. 3: Teil 1, *Theorie der Ebbe und Flut und Abhandlungen zur mathematischen Physik,* 353 pp., 1911 (herausgegeben von JUSTUS GRASSMANN und FRIEDRICH ENGEL); Teil 2, *Grassmanns Leben,* xiii + 400 pp., 1911 (geschildert von FRIEDRICH ENGEL). Leipzig, B. G. Teubner.

GRASSMANN'S works on mathematics and physics are finished after some twenty years of editorial labor.

Although Grassmann's style was such as to repel readers from the two Ausdehnungslehren, those works have slowly penetrated into the mathematical consciousness of at least a few persons, but the first part of Volume III of his works contains material that is now printed for the first time and, though written fifty or seventy-five years ago, has only now an opportunity to be valuable to the world at large. The world of science has meantime moved far on, and about the only interest in this early work must be historical, not for the history of science, merely for the history of Grassmann.

The most striking thing about the memoir Ebbe und Flut,

which was finished by April, 1840, is the great development which Grassmann had already given to his system of geometric analysis. Here we find not only vector addition and subtraction, the scalar and vector products, differentiation by a scalar, and an incisive treatment of mechanical theory by them, but we find also divergence, the linear vector function, elliptic harmonic motion and the integration of vector differential equations, and rotary operators. The work further shows that at thirty years of age Grassmann, with practically no help, had made himself master of Lagrange and, to a considerable extent at least, of Laplace, so that he could improve on both.

The mathematical physics in this volume is from the Nachlass, and is chiefly analytical optics, some of it more like Gibbs's treatment in his lectures than anything we have seen in print.

For composing the life of Grassmann, Engel had at his disposal a number of documents written by Grassmann himself or by members of his family. It was therefore possible to begin the sketch away back in 1634 (!) and to offer a detailed treatment of the early years of this very busy genius. It is interesting to see how systematically he went to work at the university to broaden his knowledge and render it fundamental in a number of subjects. We may therefore be able a little to understand how Grassmann, who started as a theologian, could have been inventing his analysis, writing on physics, composing class texts for the study of German, Latin, and mathematics, editing a political paper, a missionary paper, investigating phonetic laws, writing a dictionary to the Rig-Veda, publishing a translation of the Rig-Veda in verse, and harmonizing folk songs in three voices, at various times of his life and frequently at the same time—in addition to carrying on successfully his regular work as a teacher and administrator and bringing up nine of his eleven children.

Engel's life of Grassmann is written in a sound critical spirit, there is neither laudation nor condemnation of its subject, merely a connected and sympathetic history of him, from which the reader may get instruction and interest and inspiration. Few biographers in science have had a harder task, for few scientists have had a wider range of activity than Grassmann. We should all admit our deep obligations to Engel as editor and biographer.

E. B. Wilson.

NOTES.

The twenty-second annual meeting of the American Mathematical Society will be held in New York City on Monday and Tuesday, December 27–28, 1915. Titles and abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary by December 11. Abstracts intended to be printed in advance of the meeting should be sent in by December 4.

The thirty-sixth regular meeting of the Chicago Section, being the fifth western meeting of the Society, will be held at Columbus, Ohio, on Thursday, Friday, and Saturday, December 30–31 and January 1, in affiliation with the American association for the advancement of science. The first session will be a joint meeting with Section A of the Association, at which Professor H. S. White will deliver his retiring address as vice-president of Section A, on "Poncelet polygons," and Professor E. J. Wilczynski will deliver his retiring address as chairman of the Chicago Section. Titles and abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary of the Chicago Section by December 4.

A new edition of the List of Officers and Members of the Society is now in preparation and will be issued in January. Blanks for furnishing information have been sent to the members. A prompt response will contribute materially to the correctness and completeness of the List.

The sixty-eighth meeting of the American association for the advancement of science will be held at Columbus, Ohio, December 27 to January 1, under the presidency of Professor W. W. Campbell, of Lick Observatory. Professor A. O. Leuschner is vice-president and Professor F. R. Moulton secretary of Section A.

The concluding (October) number of volume 16 of the *Transactions of the American Mathematical Society* contains the following papers: "A type of singular points for a transformation of three variables," by W. V. Lovitt; "The re-

duction of multiple L-integrals of separated functions to iterated L-integrals," by J. K. LAMOND; "Independent generators of a group of finite order," by G. A. MILLER; "On the zeros of the function $P(x)$, complementary to the incomplete gamma function," by C. N. HASKINS; "Group properties of the residue classes of certain Kronecker modular systems and some related generalizations in number theory," by E. KIRCHER; "Sur l'intégrale de Lebesgue," by C. DE LA VALLÉE POUSSIN; "A new development of the theory of algebraic numbers," by G. E. WAHLIN; "Ruled surfaces whose flecnode curves have plane branches," by A. F. CARPENTER.

THE closing (October) number of volume 37 of the *American Journal of Mathematics* contains the following papers: "Geometrical and invariantive theory of quartic curves modulo 2," by L. E. DICKSON; "On the solutions of linear non-homogeneous partial differential equations," by L. L. STEIMLEY; "A method in the calculus of variations," by R. B. ROBBINS; "On the conformal geometry of analytic arcs," by G. A. PFEIFFER; "The non-homogeneous differential equation of parabolic type," by G. C. EVANS; "On properties of the solutions of linear q-difference equations with entire function coefficients," by T. E. MASON; "On rational sextic surfaces having a nodal curve of order 9," by C. H. SISAM.

THE annual list of American doctorates published in *Science* presents, for the academic year 1914–1915, 556 names, of which 309 are credited to the sciences. The following 25 successful candidates offered mathematics as major subject (the titles of the theses are appended): A. A. BENNETT, Princeton, "An algebraic treatment of the theorem of closure"; J. W. CAMPBELL, Chicago, "Periodic solutions of the problem of three bodies in three dimensions"; C. R. DINES, Chicago, "Functions of positive type and related topics in general analysis"; C. H. FORSYTH, Michigan, "Vital and monetary losses in the United States due to preventable deaths"; M. G. GABA, Chicago, "A set of postulates for general projective geometry of n dimensions"; OLIVE C. HAZLETT, Chicago, "On the classification and invariantive characterization of nilpotent algebras"; H. B. HEDRICK, Yale, "Some principles and processes in the construction of mathematical tables";

L. A. Hopkins, Chicago, "On the theory of the motion of the small planets with a periodic orbit for the Hilda type"; H. R. Kingston, Chicago, "Metric properties of nets of plane curves"; W. V. Lovitt, Chicago, "A type of singular points for a transformation of three variables"; W. E. Milne, Harvard, "On the degree of convergence of Birkhoff's series"; G. A. Pfeiffer, Columbia, "On the conformal geometry of analytic arcs"; V. C. Poor, Chicago, "A certain type of exact solution of the equations of motion of a viscous liquid"; H. F. Price, Pennsylvania, "Fundamental regions for certain finite groups in two complex variables"; L. J. Reed, Pennsylvania, "Some fundamental systems of formal modular invariants and covariants"; P. R. Rider, Yale, "An extension of Bliss's form of the problem of the calculus of variations, with applications to the generalization of angle"; J. Rosenbaum, Cornell, "On mixed linear integral equations over a two-dimensional region"; G. Rutledge, Illinois, "The number of abelian subgroups of groups whose orders are the powers of primes"; Caroline E. Seely, Columbia, "Certain non-linear integral equations"; C. P. Sousley, Johns Hopkins, "Invariants and covariants of the Cremona hexahedral form of the cubic surface"; Eula A. Weeks, Missouri, "A symmetrical generalization of the theory of functions"; C. J. West, Cornell, "On certain formulas for representing statistical data"; C. E. Wilder, Harvard, "Problems in the theory of ordinary linear differential equations with auxiliary conditions at more than two points"; F. B. Wiley, Chicago, "Proof of the finiteness of the modular covariants of a system of binary forms and cogredient points"; L. T. Wilson, Harvard, "Conformal transformation of curvilinear angles."

The United States Bureau of education has recently issued the following two bulletins: No. 35, on "Mathematics in the lower and middle commercial and industrial schools of various countries represented in the International commission on the teaching of mathematics," prepared by Dr. E. H. Taylor with the editorial cooperation of the American members of the Commission; No. 39, on "The training of elementary school teachers in mathematics in the countries represented by the Commission," by Dr. I. L. Kandel. These bulletins may be obtained by teachers of mathematics on application to the Bureau.

The English translation, by Professor E. R. Hedrick, of the revised edition of the second volume of Goursat's Mathematical Analysis is expected to appear in January, 1916.

The Joint committee on standards for graphic presentation, composed of representatives of a number of national organizations, has issued a preliminary report as a basis for suggestions to the committee. Copies can be obtained from the American society of mechanical engineers, 29 West 39th Street, New York City, at 10 cents each.

At the University of Kansas, Dr. C. H. Ashton has been promoted from an associate professorship to a full professorship of mathematics.

Dr. E. C. Colpitts, of the State College of Washington, has been promoted from an assistant professorship to an associate professorship of mathematics.

Miss Gertrude I. McCain has been appointed professor of mathematics in the Western College for Women, Oxford, Ohio.

Professor E. H. Jones, of Daniel Baker College, has been appointed associate professor of mathematics in the Southern Methodist University.

At Williams College Dr. H. L. Agard has been promoted to an assistant professorship of mathematics.

Dr. S. D. Killam, of the University of Alberta, has been promoted to an assistant professorship of applied mathematics.

The following appointments to instructorships in mathematics are announced: Dr. M. G. Gaba, at Cornell University; Dr. L. M. Kells, at the College of the City of New York; Mr. J. S. Mikesh, at the University of Minnesota; Dr. P. R. Rider, at the Sheffield Scientific School.

Book catalogues: The Macmillan Company, 66 Fifth Avenue, New York City, catalogue of mathematical and

astronomical books, 1915–1916.—George Gregory, 5 Argyle Street, Bath, England, catalogue no. 240–241, higher mathematics, 38 items.—Galloway and Porter, Cambridge, England, recent mathematical acquisitions, 16 items.—Arthur H. Clark Company, Caxton Building, Cleveland, Ohio, early and rare books, 27 items.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ARCHIMEDES. Opera omnia cum commentariis Eutocii. Iterum J. L. Heiberg. Vol. III. Leipzig, Teubner, 1915. 8vo. 98+448 pp. M. 9.00

AUTONNE (L.). Sur les matrices hypo-hermitiennes et sur les matrices unitaires. Paris, Gauthier-Villars, 1915. 4+78 pp. Fr.5.00

BACHMANN (P.). See ENCYCLOPÉDIE.

BIRKELAND (K.) et SKOLEM (T.). Une méthode énumérative de la géométrie. Christiania (Vid. Selsk. Skrift.), 1915. Gr. 8vo. 61 pp.

CAHEN (E.). See ENCYCLOPÉDIE.

CANTOR (G.). Contributions to the founding of the theory of transfinite numbers. Translated by P. E. B. Jourdain. Chicago and London, Open Court, 1915. 8vo. 10+211 pp. Cloth. $1.25

DEMORGAN (A.). A budget of paradoxes. 2d edition, edited by D. E. Smith. 2 volumes. Chicago, Open Court, 1915. 8vo. 8+402+387 pp. Cloth. $7.00

DICKSON (L. E.). Algebraic invariants. (Mathematical monograph series, No. 14.) New York, Wiley, 1915. 8vo. $1.25

DINGELDEY (F.). See SALMON (G.)

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome I, volume 3, fascicule 5: Propositions transcendantes de la théorie des nombres, par P. Bachmann, J. Hadamard et E. Maillet; Théorie des corps de nombres algébriques, par D. Hilbert et H. Vogt; Multiplication complexe, par H. Weber et E. Cahen. Leipzig, Teubner, 1915. Gr. 8vo. Pp. 385–480. M. 3.00

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III 3, Heft 4: H. Liebmann, Berührungstransformationen; Geometrische Theorie der Differentialgleichungen. Leipzig, Teubner, 1915. Gr. 8vo. Pp. 441–539. M. 3.20

ENGEL (F.). Zur Differentialgeometrie der komplexen analytischen Flächen. Strassburg, 1914. 8vo. 37 pp.

FIEDLER (W.). See SALMON (G.).

GOLDENRING (R.). Die elementargeometrischen Konstruktionen des regelmässigen Siebzehnecks. Eine historisch-kritische Darstellung. Leipzig, Teubner, 1915. Gr. 8vo. 7+69 pp. Geh. M. 2.80

Goursat (E.). Cours d'analyse mathématique. 2e édition, entièrement refondu. Tome 3: Intégrales infiniment voisines. Equations aux dérivées partielles du second ordre. Equations intégrales. Calcul des variations. Paris, Gauthier-Villars, 1915. Gr. 8vo. 667 pp. Fr. 20.00

Guimaraes (R.). Sur la vie et l'œuvre de Pedro Nuñes. Coïmbre, Imprimerie de l'Université, 1915. 8vo. 87 pp.

Gumppenberg (H. v.). Beweis des grossen Fermatschen Satzes für alle ungeraden Exponenten. München, 1915. Gr. 8vo. 124 pp. M. 2.80

Hadamard (J.). See Encyclopédie.

Heiberg (J. L.). See Archimedes.

Hilbert (D.). See Encyclopédie.

Jordi (E.). Ueber Reihenentwicklungen nach Quadraten und Produkten von Besselschen Funktionen. Bern, 1913. 8vo. 111 pp.

Jourdain (P. E. B.). See Cantor (G.).

Kaye (G. R.). Indian mathematics. Calcutta and Simla, Thacker, Spink and Co., 1915. 73 pp.

Kloosterboer (G. W.). Coördinatentafel. Sinus- en cosinustafels ter berekening von rechthoekige coördinaten. Deventer, 1914. Fol. Geb. M. 4.80

Landis (E. H.). See Richardson (R. P.).

Liebmann (H.). See Encyklopädie.

Lorentz (H. A.). Lehrbuch der Differential- und Integralrechnung nebst Einführung in andere Teile der Mathematik, mit besonderer Berücksichtigung der Bedürfnisse der Studierenden der Naturwissenschaften. Uebersetzt von G. C. Schmidt. 3te Auflage. Leipzig, 1915. 8vo. 7+602 pp. M. 14.00

Maillet (E.). See Encyclopédie.

Nalli (P.). Esposizione e confronto critico delle diverse definizioni proposte per l'integrale definito di una funzione limitata o no. Palermo, Virzì, 1914. 162 pp.

Richardson (R. P.) and Landis (E. H.). Fundamental conceptions of modern mathematics. Chicago, Open Court, 1915. $1.25

Salmon (G.) und Fielder (W.). Analytische Geometrie der Kegelschnitte. Neu herausgegeben von F. Dingeldey. In 2 Teilen. 1ter Teil. 8te Auflage. Leipzig, Teubner, 1915. Gr. 8vo. 30+452 pp. Geb. M. 12.00

Schmidt (G. C.). See Lorentz (H. A.).

Skolem (T.). See Birkeland (K.).

Smith (D. E.). See DeMorgan (A.).

Stuckenberg (K.). Elliptische Wurzelfunktionen. Strassburg, 1913. 8vo. 57 pp.

Vogt (H.). See Encyclopédie.

Weber (H.). See Encyclopédie.

II. ELEMENTARY MATHEMATICS.

Bibliography of education for 1911–12. (United States Bureau of Education Bulletin No. 657). Washington, Government Printing Office, 1915. 151 pp. $0.20

Breslich (E. R.). First-year mathematics for secondary schools. 3d edition. Chicago, University of Chicago Press, 1915. 8vo. 26+344 pp. Cloth. $1.00

Carcano (C.). Matematica semplificata pratica, per scuole commerciali, industriali, licei, istituti tecnici, capomastri, ingegneri, professionisti; numerosi esercizi pratici completamente risolti. Vol. 3 (Progessioni geometriche). Milano, tip. Pivola e Cella, di P. Cella, 1915. 16mo. 63 pp. L. 1.00

Carson (G. St. L.) and Smith (D. E.). Elements of algebra. Part 2. London, Ginn, 1915. Pp. 5+325–538. 2s. 6d.

——. Plane Geometry. 2 parts. London, Ginn, 1915. 6+266+6+224 pp. 2s. 6d.+2s. 6d.

Giebel (K.). Die Anfertigung mathematischer Modelle für Schüler mittlerer Klassen. (Mathematische Bibliothek No. 16). Leipzig, Teubner, 1915. 8vo. 4+52 pp. Boards. M. 0.80

Godfrey (C.) and Price (E. A.). Arithmetic. Parts 1–3 complete, with answers. Cambridge, University Press, 1915. 13+467 pp. 4s.

Hadamard (J.). Leçons de géométrie élémentaire. I. Géométrie élémentaire. 5e édition. Paris, Colin, 1913.

Könnemann (W.). Rationale Lösungen von Aufgaben aus dem Gebiete der gesamten Elementarmathematik in funktioneller Abhängigkeit. Berlin, Winckelmann, 1915. 8vo. Geb. 8+112 pp. M. 3.00

Lennes (N. J.). See Slaught (H. E.).

M'Laren (G. C.). Improved four-place logarithm-tables. Multiplication and divison made easy. Cambridge, University Press, 1915. 27 pp. 1s. 6d.

Mancinelli (F.). Alcune osservazioni di trigonometria piana. Forma probabile del triangolo e del quadrangolo. Problema fondamentale della planimetria. Pavia, tip. Popolare, di P. Mozzaglia, 1915. 8vo. 7+4+7 pp.

Merriman (M.). Mathematical tables for class-room use. New York, Wiley, 1915. 8vo. 68 pp. $0.50

Monroe (P.). See Smith (D. E.).

Norman (F. K.). See Norman (J. S.).

Norman (J. S. and F. K.). Norman's arithmetic for schools. London, Year Book Press, 1915. 8vo. 16+278 pp. 2s. With answers, 2s. 6d.

Price (E. A.). See Godfrey (C.).

Reed (H. L.). Plane trigonometry. London, Bell, 1915. 13+290+16 pp. 3s. 6d.

Slaught (H. E.) and Lennes (N. J.). Elementary algebra. Boston, Allyn and Bacon, 1915. 8vo. 10+357 pp. $1.00

SMITH (D. E.). Problems about war: for classes in arithmetic. With an introduction by Paul Monroe. New York, The Carnegie Endowment for International Peace, 1915. 23 pp.

——. See CARSON (G. ST. L.).

III. APPLIED MATHEMATICS.

ANDREWS (E. S.). An introduction to applied mechanics. Cambridge University Press, 1915. 8vo. 4s. 6d.

ANNUAIRE pour l'an 1915 publié par le Bureau des longitudes avec une notice scientifique par M. G. Bigourdain. Paris, Gauthier-Villars, 1915. 7+764+173+58 pp. Fr. 1.50

ANSEL (E. A.). Beiträge zur Dynamik und Thermodynamik der Atmosphäre. Göttingen, 1913. 8vo. 67 pp.+4 Tafeln. M. 2.00

BIGOURDAIN (G.). See ANNUAIRE.

BLOCK (E.). Ueber die Schmelzkurven einiger Stoffe. Göttingen, 1913. 8vo. 40 pp.

BOLZA (H.). Anwendung der Theorie der Integralgleichungen auf die Elektronentheorie und die Theorie der verdünnten Gase. Göttingen, 1913. 8vo. 36 pp.

BRUNNER (W.). Dreht sich die Erde? (Mathematische Bibliothek No. 17.) Leipzig, Teubner, 1915. Klein 8vo. 4+52 pp. Boards. M. 0.80

BURNHAM (R. W.). Mathematics for machinists. New York, Wiley, 1915. 8vo. 7+229 pp. $1.25

CASPAR (E.). Neue Methode zur Messung kleiner Temperaturdifferenzen, angewandt auf die Untersuchung der Temperaturverhältnisse in Kältemischungen mit Kohlensäureschnee. Marburg, 1913. 8vo. 68 pp.+1 Tafel.

CHAFFEE (E. L.). Physical laboratory manual. Cambridge, Mass., 1914. 8vo. 128 pp. $1.50

CLIFFORD (W. G.). See REECE (T.).

CRAPPER (E. H.). Arithmetic of alternating currents. London, Whittaker, 1915. 7+208 pp. 2s. 6d.

DALE (R. B.). Arithmetic for carpenters and builders. New York, Wiley, 1915. 8vo. 6+228 pp. $1.25

DARWIN (E.). Emma Darwin: a century of family letters, 1792–1896. Edited by her daughter Henrietta Litchfield. 2 volumes. London, Murray, 1915. 31+289+25+326 pp. 21s.

DONNER (G.). Ueber die Stabilität der quasi-beständigen Bewegung. Strassburg, 1913. 8vo. 44 pp.

DUCHÊNE (—.). The mechanics of the aeroplane, by Captain Duchêne. Translated from the French by J. H. Ledeboer and T. O'B. Hubbard. New edition. New York, Longmans, 1915. 10+230 pp. $2.25

DUHEM (P.). La science allemande. Paris, Hermann, 1915. 146 pp.

EGGERT (O.). See JORDAN (W.).

FAWDRY (R. C.). Statics. Part 2. London, Bell, 1915. Pp. 5+159–305+8. 2 s.

HAAG (H.). Die Geschichte des Nullmeridians. Giessen, 1913. 8vo. 111 pp.+1 Karte.

HENDERSON (R.). Mortality laws and statistics. (Mathematical monograph series, No. 15.) New York, Wiley, 1915. 8vo. 6+111 pp. Cloth. $1.25

HUBBARD (T. O'B.). See DUCHÊNE (—.).

JAHNKE (E.). See RUNGE (C.).

JOHANSEN (N. P.). Astronomisk Bestemmelse af Laengdedifferenzen mellem Kjöbenhavns Observatorium og Buddinge samt af Azimuthes i Buddinge af Retningen mod Nikolaj Taarn. Kjöbenhavn (Danske Gradmaal), 1914. 4to. 120 pp. M. 5.00

JOHNSON (J. F.). Practical shop mechanics and mathematics. New York, Wiley, 1915. 8vo. 7+130 pp. $1.00

JONES (H. S.). Numerical examples in physics. London, Bell, 1915. 8vo. 12+332 pp. 3s. 6d.

JORDAN (W.). Handbuch der Vermessungskunde von W. Jordan fortgesetzt von C. Rheinhertz. Band 2: Feld- und Landmessung. 8te erweiterte Auflage bearbeitet von O. Eggert. Stuttgart, J. B. Metzler, 1914. 8vo. 10+938+55 pp. M. 22.50

JOURDAIN (P. E. B.). See MACH (E.).

KRÖNCKE (H.). Ueber die Messung der Intensität und Härte der Röntgenstrahlen. Göttingen, 1913. 8vo. 55 pp.

LECORNU (L.). Cours de mécanique professé à l'Ecole polytechnique. Tome 2. Paris, Gauthier-Villars, 1915. 6+538 pp. Fr. 18.00

LEDEBOER (J. H.). See DUCHÊNE (—.).

LILLY (S. B.). See MILLER (J. A.).

LITCHFIELD (H.). See DARWIN (E.).

LUCY (A. W.). Exercises in laboratory mathematics. Oxford, Clarendon Press, 1915. 245 pp. 3s. 6d.

LYNDE (C. J.). Physics of the household. New York, Macmillan, 1915. 8vo. 315 pp. 5s. 6d.

MACH (E.). The science of mechanics: a critical and historical account of its development. Supplement to the third English edition translated and annotated by P. E. B. Jourdain. London and Chicago, Open Court, 1915. 14+106 pp. 2s. 6d.

MANN (H. L.). A text-book on practical mathematics for advanced technical students. London, Longmans, 1915. 12+488 pp. 7s. 6d.

MILLER (H. W.). Descriptive geometry. 3d edition. New York, Wiley, 1915. 8vo. 149 pp. $1.50

MILLER (J. A.) and LILLY (S. B.). Analytic mechanics. Boston and New York, Heath, 1915. 15+297 pp. $2.00

PARKER (G. W.). Elements of optics for the use of schools and colleges. London and New York, Longmans, 1915. 8vo. 122 pp. $0.75

REECE (T.) and CLIFFORD (W. G.). Billiards. London, A. and C. Black, 1915. 8vo. 7s. 6d.

RHEINHERTZ (C.). See JORDAN (W.).

RUNGE (C.). Graphische Methoden. Uebersetzung der Vorlesungen, die im Winter 1909–10 an der Columbia-Universität gehalten waren. (Sammlung mathematisch-physikalischer Lehrbücher herausgegeben von E. Jahnke, Band 18.) Leipzig, Teubner, 1915. 8vo. 4+142 pp. Geb. M. 5.00

SCHUR (F.). Vorlesungen über graphische Statik. Unter Mitwirkung von W. Vogt. Leipzig, 1915. Gr. 8vo. 8+219 pp. M. 7.00

STARK (J.). Prinzipien der Atomdynamik. Teil 3: Die Elektrizität im chemischen Atom. Leipzig, 1915. 8vo. 16+280 pp. M. 8.00

TRAUB (G.). Ueber die Vertical-Geschwindigkeitskurve. Untersuchungen über die Verteilung der Wassergeschwindigkeiten im offenen Wasserlauf. Dresden, 1913. 8vo. 160 pp.+6 Tafeln. M. 4.00

VALIER (M.). Das astronomische Zeichnen. Leichtfassliche und gemeinverständliche Anleitung zur Beobachtung und zeichnerischen Darstellung cölestischen Objekte nach dem Anblick im Fernrohr für Laien und Amaturastronomen. Mit Anhang: Mondaufnahmen. München, 1915. 8vo. M. 1.50

VOGT (W.). See SCHUR (F.).

THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The one hundred and seventy-ninth regular meeting of the Society was held in New York City on Saturday, October 30, 1915. The attendance at the morning and afternoon sessions included the following fifty members:

Professor R. C. Archibald, Dr. A. A. Bennett, Mr. D. R. Belcher, Professor E. G. Bill, Professor W. J. Berry, Professor G. D. Birkhoff, Professor E. W. Brown, Dr. T. H. Brown, Dr. Emily Coddington, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Dr. H. B. Curtis, Mrs. E. B. Davis, Dr. C. R. Dines, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor W. B. Fite, Professor O. E. Glenn, Dr. G. M. Green, Professor C. N. Haskins, Professor H. E. Hawkes, Professor L. A. Howland, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. P. H. Linehan, Dr. H. F. MacNeish, Dr. L. C. Mathewson; Professor F. M. Morgan; Mr. George Paaswell, Dr. G. A. Pfeiffer, Professor H. W. Reddick, Professor L. W. Reid, Professor R. G. D. Richardson, Mr. J. F. Ritt, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Professor W. B. Stone, Mr. H. S. Vandiver, Professor Oswald Veblen, Dr. Mary E. Wells, Professor H. S. White, Miss E. C. Williams, Professor A. H. Wilson, Professor J. W. Young.

The President of the Society, Professor E. W. Brown, occupied the chair, being relieved by Vice-President Oswald Veblen. The Council announced the election of the following persons to membership in the Society: Mr. D. R. Belcher, Columbia University; Professor J. W. Calhoun, University of Texas; Professor Sarah E. Cronin, State University of Iowa; Mr. C. A. Epperson, State Normal School, Kirksville, Mo.; Dr. Olive C. Hazlett, Radcliffe College; Mr. C. M. Hebbert, University of Illinois; Miss Goldie P. Horton, University of Texas; Professor W. S. Lake, Bendigo School of Mines and Industries, Australia; Mr. D. H. Leavens, College of Yale in China; Mr. C. T. Levy, University of California; Dr. F. W. Reed, University of Illinois; Professor L. H. Rice, Syracuse University; Mr. J. F. Ritt, Columbia University; Professor

D. M. Y. Sommerville, Victoria University College, Wellington, N. Z.; Miss L. R. Stoughton, Rosemary Hall School, Greenwich, Conn.; Dr. C. E. Wilder, Pennsylvania State College; Mr. A. R. Williams, University of California; Dr. L. T. Wilson, University of Illinois; Dr. F. E. Wright, U. S. Geological Survey. Four applications for membership in the Society were received.

The Council submitted a list of nominations for officers and other members of the Council, to be placed on the official ballot for the annual election. A committee was appointed to audit the accounts of the Treasurer for the current year.

The dinner in the evening, always a pleasant feature of the meetings, was attended by twenty-one members and friends.

The following papers were read at this meeting:

(1) Dr. G. A. Pfeiffer: "Existence of divergent solutions of the functional equations $\varphi[g(x)] = a\varphi(x)$, $f[f(x)] = g(x)$, where $g(x)$ is a given analytic function, in the irrational case."

(2) Professor C. N. Haskins: "On the extremes of bounded summable functions and the distribution of their functional values."

(3) Dr. G. M. Green: "Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface. Second memoir."

(4) Dr. G. M. Green: "The linear dependence of functions of several variables."

(5) Mr. A. R. Schweitzer: "On the dependence of algebraic equations upon quasi-transitiveness."

(6) Professor H. S. Carslaw: "A trigonometrical sum and the Gibbs phenomenon in Fourier's series."

(7) Professor W. F. Osgood: "On a sufficient condition for a non-essential singularity of a function of several complex variables."

(8) Dr. Dunham Jackson: "Singular points of functions of several complex variables."

(9) Professor W. F. Osgood: "On functions of several complex variables."

(10) Professor L. P. Eisenhart: "Envelopes of rolling and transformations of Ribaucour."

(11) Professor W. B. Fite: "Note on linear homogeneous differential equations of the second order."

(12) Mr. H. S. Vandiver: "Note on the distribution of quadratic residues."

(13) Professor G. D. BIRKHOFF: "A theorem concerning the singular points of ordinary linear differential equations."

(14) Professor H. S. WHITE: "Closed systems of sevens in a 3–3 correspondence."

(15) Professor W. R. LONGLEY: "Note on a theorem on envelopes."

(16) Mr. A. R. SCHWEITZER: "On the dependence of algebraic equations upon quasi-transitiveness. Second paper."

(17) Mr. A. R. SCHWEITZER: "A new functional characterization of the arithmetic mean."

Professor Carslaw's paper was communicated to the Society through Professor Bôcher. In the absence of the authors Professor Osgood's second paper was read by Professor Birkhoff and the papers of Mr. Schweitzer, Professor Carslaw, Professor Longley, and the first paper of Professor Osgood were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The first functional equation considered in Dr. Pfeiffer's paper arises in the consideration of the conformal equivalence of curvilinear angles in the so called irrational case. In this case a transformation can always be found formally. The author shows that the series giving the transformation may be divergent. The second functional equation here considered determines the bisectors of an irrational curvilinear angle as defined by Kasner, and it is proved that such an angle may have two, one, or no bisectors.

Stated specifically the theorems obtained are

1. There exists an analytic function $g(x) \equiv a_1x + a_2x^2 + \cdots$ ($|a_1| = 1,\ a_1^n \neq 1,\ n = 1, 2, \cdots$) such that every formal solution $\phi(x) = c_1x + c_2x^2 + \cdots$ ($c_1 \neq 0$) of the functional equation $\phi[g(x)] = a_1\phi(x)$ is divergent for all values of $x \neq 0$.

2. Given the functional equation $f[f(x)] = g(x)$. Then $g(x) \equiv a_1x + a_2x^2 + \cdots$ ($|a_1| = 1,\ a_1^n \neq 1,\ n = 1, 2, \cdots$) may be taken such that the given equation has (a) no solution analytic about the origin, (b) only one such solution, (c) two such solutions. The number of solutions cannot be greater than two.

The author is indebted to Professor Birkhoff for suggestions which led to the above development.

2. Professor Haskins's paper presents, first, a method of the integral calculus, as contrasted with the classic methods of the differential calculus, for the determination of the extrema extremorum of bounded Lebesgue-integrable functions; and second, certain results concerning what may be termed the statistical distribution of functional values.

The methods used are allied to those of Laplace-Darboux-Stieltjes for the determination of "functions of large numbers." Some of the results (for the case of continuous functions and Riemann integrals) have already been presented to the Society. The full significance of the results appears however only when the Lebesgue integral is used, for the reason that these results relate to certain constants and certain analytic functions which serve to divide all Lebesgue-integrable functions into classes such that all the functions of each class and only those have the same defining elements of their Lebesgue integrals.

3. In a recent paper,* Dr. Green laid the foundations for a purely projective theory of conjugate nets on a curved surface, his present paper forming a continuation of this study. The first part of the memoir contains a canonical development of the non-homogeneous coordinates of the surface in the neighborhood of a point, which was the subject of a previous communication to the Society.† The discussion is completed in the present paper. The tetrahedron of reference which gave rise to the canonical development has as two of its edges lines which Wilczynski‡ has called the axis and the ray of a point of the surface. The second part of Dr. Green's paper contains an investigation of the properties of the conjugate net expressed in terms of the axis and ray congruences. A new net of curves is introduced, viz., the associate conjugate net, the tangents to the two curves of which at any point separate harmonically the tangents to the curves of the original conjugate net. Any conjugate net has one and only one associate conjugate net. It appears that the theory of a conjugate net is very conveniently investigated through the consideration thereof in connection with its associate conjugate net. A discussion from this point of view leads to many interesting theorems, some of

---

* *Amer. Jour. of Mathematics*, vol. 37 (1915), pp. 215–246.

† See this BULLETIN, vol. 20, no. 8 (May, 1914), p. 397.

‡ *Transactions*, vol. 16 (1915), pp. 311–327.

them already known; in fact, the study of a conjugate net, its associate conjugate net, and the axis and ray congruences for both, seems to afford a powerful method for describing geometrically projective properties of the conjugate net. An application is made at the end of the paper to the conjugate nets which Bianchi has called isothermally conjugate. For such a net, for instance, the associate conjugate net is also isothermally conjugate.

4. For a set of functions of a single real variable, it is well known that in certain cases the vanishing of the wronskian is a sufficient condition for linear dependence. The wronskian arises naturally out of the study of a single ordinary homogeneous linear differential equation, the characteristic property of which is that any solution is expressible linearly, with constant coefficients, in terms of a fundamental set of solutions. The natural generalization of this kind of ordinary differential equation is the completely integrable system of homogeneous linear partial differential equations, the solutions of which have the same property. A year ago, Dr. Green* gave a generalization for such systems of the notion of wronskian, but this can be done only when the completely integrable system is given. It is the object of Dr. Green's second paper to supply a sufficient condition for the linear dependence of a set of functions of several independent real variables. There is no determinant of fixed form analogous to the wronskian, but the criteria given are very general, including by specialization those concerning the wronskian for functions of a single variable. Application of the theorems is made to completely integrable systems, yielding results analogous to those for ordinary homogeneous linear differential equations.

5. In the first part of Mr. Schweitzer's paper it is shown that if

$$(1)\quad f(x_1 + y_1, x_2 + y_2, \cdots, x_{n+1} + y_{n+1}) = f(x_1, x_2, \cdots, x_{n+1}) + f(y_1, y_2, \cdots, y_{n+1}),$$

$$(2)\quad f\{f(t_1, t_2, \cdots, t_n, x_1)f(t_1, t_2, \cdots, t_n, x_2) \cdots f(t_1, t_2, \cdots, t_n, x_{n+1})\} = f\left(\frac{x_2}{c_1}, \frac{x_3}{c_2}, \cdots, \frac{x_{n+1}}{c_n}, \frac{x_1}{c_{n+1}}\right),$$

* See this BULLETIN, vol. 21, no. 4 (Jan., 1915), p. 162.

then

$$f(x_1, x_2, \cdots, x_{n+1}) = \sum_{i=1}^{n} [c'_{n-i+1}\xi^{n-i+2} \cdot x_i] + \xi x_{n+1},$$

where

$$c'_{n-i+1} = c_i \cdot c_{i+1} \cdots c_n, \qquad \frac{1}{c_{n+1}} = c_1 \cdot c_2 \cdots c_n \cdot \xi^{n+1},$$

$$1 + \sum_{i=1}^{n} c'_{n-i+1} \cdot \xi^{n-i+1} = 0.$$

In the second part of the paper an alternative method of inducing algebraic equations of the nth degree is discussed. The latter treatment is based on the equation (1) in conjunction with functional equations derived from the relation

$$f\{f_1(t_1^{(1)}, t_2^{(1)}, \cdots, t_n^{(1)}, x_1), f_2(t_1^{(2)}, t_2^{(2)}, \cdots, t_n^{(2)}, x_2) \cdots f_{n+1}(t_1^{(n+1)}, t_2^{(n+1)}, \cdots, t_n^{(n+1)}, x_{n+1})\} = \psi(x_1, x_2, \cdots, x_{n+1})$$

(or analogous relations obtained by the homologous transposition of the x's on the left side) by substituting suitable x's for some or all of the t's and assuming the remaining t's (if any) with the same subscript to be identical. In the simplest cases the following theorems are valid:

I. $f\{f(x_3, x_1, t), f(x_1, x_2, t), f(x_2, x_3, t)\} = f(x_2, x_1, x_3)$ implies

$$f(x_1, x_2, x_3) = \frac{1}{2}x_1 + \frac{1 \pm \sqrt{13}}{4}x_2 - \frac{3 \pm \sqrt{13}}{4}x_3.$$

II. $f\{f(x_2, x_3, x_1), f(x_3, x_1, x_2), f(x_1, x_2, x_3)\} = f(x_2, x_3, x_1)$ implies

$$f(x_1, x_2, x_3) = \tfrac{1}{3}(2x_1 - cx_2 - c^2x_3) \quad \text{or} \quad \tfrac{1}{3}(x_1 + cx_2 + c^2x_3) \quad \text{or} \quad \tfrac{1}{3}(x_1 + x_2 + x_3),$$

where $1 + c + c^2 = 0$.

III. $f\{f(x_3, t, x_1)f(x_1, t, x_2)f(x_2, t, x_3)\} = f(x_2, x_1, x_3)$ implies $f(x_1, x_2, x_3) = -c^2x_1 - c^4x_2 + cx_3$, where $c^3 + c - 1 = 0$.

IV. $f\{f(t, x_3, x_1)f(t, x_1, x_2)f(t, x_2, x_3)\} = f(x_2, x_1, x_3)$ implies

$$f(x_1, x_2, x_3) = -\frac{1}{2}x_1 + \frac{1 \pm \sqrt{5}}{4}x_2 + \frac{1 \mp \sqrt{5}}{4}x_3$$

or $2x_1 - x_2 - x_3$.

In deriving the preceding results the operation of differentiation is not employed.

6. In Professor Carslaw's paper the Gibbs phenomenon in Fourier series is deduced from the behavior of the approximation curves for the Fourier series

$$2(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \cdots). \tag{1}$$

In the interval $-\pi < x < \pi$, this series represents the function defined by the following equations:

$$f(-\pi) = f(\pi) = f(0) = 0, \qquad f(x) = -\tfrac{1}{2}\pi(-\pi < x < 0),$$
$$f(x) = \tfrac{1}{2}\pi(0 < x < \pi).$$

Bôcher, in his discussion of the Gibbs phenomenon (*Annals of Mathematics*, (2), volume 7, 1906, and *Crelle's Journal*, volume 144, 1914), uses the series

$$\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \cdots. \tag{2}$$

He deals with the maxima and minima of $R_n(x)$, the remainder after n terms.

Gronwall (*Mathematische Annalen*, volume 72, 1912) discusses the question of the behavior of the maxima and minima of $S_n(x)$, the sum of n terms of the series (2), and deduces the Gibbs phenomenon from the properties of $S_n(x)$. But both his results and the method by which they are obtained are somewhat complicated.

In the present paper the behavior of the maxima and minima of $S_n(x)$ for the series (1) is examined. The properties of the approximation curves $y = S_n(x)$ are much simpler for (1) than for (2). From the results obtained, the Gibbs phenomenon for the series (1) follows immediately. The extension to the general case is simple.

7. Professor Osgood's first paper contains the following theorem. In order that a function $F(z_1, \cdots, z_n)$ have a non-essential singularity in a point $(z) = (a)$ it is sufficient:

(1) that $F(z_1, \cdots, z_n)$ be uniquely defined and analytic at all points of a certain neighborhood Σ of the point (a), which do not lie on a locus defined by an equation of the form $H(z_1, \cdots, z_n) = 0$, where $H(z_1, \cdots, z_n)$ is analytic in the point (a) and vanishes there, but does not vanish identically.

(2) that $F(z_1, \cdots, z_n)$ in general become infinite in those points of the above locus which lie in Σ, the excepted points being those which lie simultaneously on a second locus, $G(z_1, \cdots, z_n) = 0$, where $G(z_1, \cdots, z_n)$ is subject to the same conditions as $H(z_1, \cdots, z_n)$, and where furthermore G and H have no common factor in the point $(z) = (a)$.

The proof is given in the author's Funktionentheorie, volume II, chapter 3, § 2.

8. It is a familiar property of functions of a single complex variable that if $f(z)$ has a pole at the point a, it can be written throughout the neighborhood of this point in the form $f(z) = \varphi(z)/(z - a)^m$, where m is a suitable positive integer and $\varphi(z)$ is a function of z which is analytic at a and does not vanish there. Dr. Jackson gives one of the possible generalizations of this theorem to functions of several variables, differing somewhat in its hypotheses from one which has been obtained by Professor Osgood. It is assumed that $f(z_1, z_2, \cdots, z_n)$ is analytic throughout the neighborhood of $(a_1, a_2, \cdots, a_n)$, except for singularities which satisfy a certain requirement of continuity in their distribution, and which are of such a nature that f is analytic except for poles when regarded as a function of z_1 alone; and it is shown that under these conditions f can be expressed as the quotient of two functions, each analytic in the neighborhood of $(a_1, a_2, \cdots, a_n)$. The denominator is of course no longer merely a power of a linear factor, in general, and the numerator may vanish at points of the neighborhood in question. The proof is closely related in method to well-known proofs of Weierstrass's theorem of factorization.

9. Let $x_i = \varphi_i(u_1, \cdots, u_{n-1})$, $i = 1, \cdots, n$, be a set of functions, each analytic at the origin $(u) = (0)$, and vanishing there. Will the points $(x_1, \cdots, x_n)$ of the locus thus defined lie on a hypersurface $\Omega(x_1, \cdots, x_n) = 0$, where Ω is analytic at the origin and vanishes there, but does not vanish identically? If $n = 2$, it is well known that the answer is affirmative. Professor Osgood shows by an example that, when $n > 2$, the corresponding theorem is false.

Let the functions $f_i(u_1, \cdots, u_n)$, $i = 1, \cdots, n$, be analytic at the point $(u) = (b)$, and let $f_i(b_1, \cdots, b_n) = a_i$. Let T be an arbitrary neighborhood of the point $(a) = (a_1, \cdots, a_n)$.

Finally, let the Jacobian of the functions f_i vanish identically. Then it is well known that there exists a point (a') of T and a function $\Omega(x_1, \cdots, x_n)$ analytic at (a') and not identically $= 0$, and furthermore such that, if x_i be replaced by f_i, Ω then goes over into a function of $(u_1, \cdots, u_n)$ which vanishes identically in these arguments. Can the point (a') be taken at (a)?

Professor Bliss has shown that it can when $n = 2$. It is shown in the present paper that when $n > 2$ this is not in general possible.

Thirdly, the tentative generalization of Weierstrass's theorem of factorization, concerning which Professor Osgood raised a query in the Madison Colloquium Lectures, page 185, is shown to be impossible when $n > 1$.

Finally, a function z of x and y defined by the equation $G(x, y, z) = 0$, where G is analytic at the origin and vanishes there, but does not vanish identically, may have a natural boundary in every neighborhood of the origin.

10. In a note published in volume 23 of the *Rendiconti dei Lincei* Bianchi has defined as an envelope of rolling the surface enveloped by a plane invariably fixed with respect to a surface S_0 as the latter rolls over an applicable surface S, which Bianchi calls the surface of support. He shows that, given any surface Σ, the problem of finding pairs of applicable surfaces S_0 and S such that Σ is an envelope of rolling as S_0 rolls over S reduces to the integration of a partial differential equation of the second order and to quadratures. Two surfaces are said to be in the relation of a transformation of Ribaucour when they constitute the envelope of a two-parameter family of spheres such that the lines of curvature correspond on the two surfaces, corresponding points being on the same sphere. Professor Eisenhart has shown that the necessary and sufficient condition that either surface of two so related be an envelope of rolling with the locus of centers of the spheres for surface of support is that the correspondence of the spherical representations of the lines of curvature of the two surfaces in the relation of a transformation of Ribaucour be conformal. This latter condition is satisfied in case these spherical representations are isothermal, and only in this case. Any surface with isothermal spherical representation of its lines of curvature admits of transforma-

tions of Ribaucour of this kind, as the author has shown in several former papers.

11. Professor Fite determines certain intervals within which a solution of the equation $y'' + py' + qy = 0$ and its derivative cannot both vanish. The argument is based upon the obvious fact that such a solution and its derivative cannot both vanish in any interval within which q is negative.

12. In the BULLETIN for November, 1915, Mr. Vandiver considered some theorems which he applies in the present note to problems regarding the distribution of quadratic residues for a prime modulus. Some special quadratic forms are also considered.

13. Professor Birkhoff proves that if $Y(x)$ is the matrix solution of a linear differential system with singular point of rank $q + 1$ at $x = \infty$, then, if $\varphi(x) - x$ vanishes to the qth order at $x = \infty$, the matrix $A(x)$ defined by the equation $Y(\varphi(x)) = A(x)Y(x)$ is composed of elements analytic at $x = \infty$. The converse is also proved. This note is to appear in the *Proceedings of the National Academy of Sciences.*

14. In a previous note Professor White had proved a property of seven points on a gauche cubic curve: that certain sets of seven planes determined by them osculate other cubic curves. In the present note it is established that such a set of seven points on the one curve and seven osculating planes of the other curve is variable while the curves remain fixed, every point of the first curve belonging to one such set of seven.

15. The theorem in question was published by Risley and MacDonald in the *Annals of Mathematics*, second series, volume 12, page 86, and gives certain criteria for the existence and non-existence of an envelope of the system of curves $y = f(x, \alpha)$ in the neighborhood of a point (x_0, y_0). In the present note Professor Longley points out that the conclusion of the theorem does not follow from the argument and gives some examples which show that it is impossible from the hypotheses of the theorem to draw any conclusion concerning the non-existence of an envelope in the neighborhood of the point in question.

16. In a previous communication Mr. Schweitzer defined a (symmetric) group of functional equations on $n + 1$ variables ($n = 1, 2, 3, \cdots$) viz.,

$$f\{f(t_1, t_2, \cdots, t_n, x_1), f(t_1, t_2, \cdots, t_n, x_2), \\ \cdots f(t_1, t_2, \cdots, t_n, x_{n+1})\} = f\{x_{i_1}, x_{i_2}, \cdots, x_{i_{n+1}}\}.$$

In the present paper the group of equations is studied which results from the preceding group by replacing, in the above relation, the x_{i_k} by the functions $\alpha_k(x_{i_k})$ [$k=1, 2, \cdots, (n+1)$]. In particular, the relation corresponding to the substitution [$1, 2, \cdots, (n + 1)$] implies that there exists a function $\psi(x)$ such that

$$\psi\alpha_k(x) = m_k\psi(x) + p_k,$$

$$\psi f(x_1, x_2, \cdots, x_{n+1}) = \sum_{i=1}^{n} \frac{\xi^{n-i+2} \cdot \psi(x_i)}{m_i \cdot m_{i+1} \cdots m_n} + \xi\psi(x_{n+1}) + l,$$

where ξ, m_k, p_k, l are constants such that

$$\xi^{n+1} = m_1 \cdot m_2 \cdots m_n \cdot m_{n+1}, \quad \sum_{i=1}^{n} \frac{\xi^{n-i+2}}{m_i m_{i+1} \cdots m_n} + \xi = 0,$$

and

$$\sum_{i=1}^{n} \frac{\xi^{n-i+2} p_i}{m_i m_{i+1} \cdots m_n} + \xi p_{n+1} = 0.$$

By making special assumptions concerning the functions $\alpha_k(x)$ the function $\psi(x)$ can be particularized.

17. The postulates constructed by Mr. Schweitzer for the arithmetic mean are as follows:

1. $f\{x_1 + y_1, x_2 + y_2, \cdots, x_{n+1} + y_{n+1}\}$
$$= f(x_1, x_2, \cdots, x_{n+1}) + f(y_1, y_2, \cdots, y_{n+1}).$$

2. $f(x_1, x_2, \cdots, x_{n+1}) = f(x_2, x_3, \cdots, x_{n+1}, x_1).$

3. $f\{f(x_1, x_2, \cdots, x_n, x_{n+1}), f(x_2, x_3, \cdots, x_{n+1}, x_1),$
$$\cdots f(x_{n+1}, x_1, x_2, \cdots, x_n)\} = f(x_1, x_2, \cdots, x_n, x_{n+1}).$$

F. N. Cole,
Secretary.

THE TWENTY-SEVENTH REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-seventh regular meeting of the San Francisco Section of the Society was held at Stanford University on November 20, 1915. Sixteen persons were present, including the following members of the Society:

Professor R. E. Allardice, Dr. B. A. Bernstein, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor G. C. Edwards, Professor M. W. Haskell, Professor L. M. Hoskins, Dr. Frank Irwin, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Professor E. W. Ponzer, Professor T. M. Putnam.

The chairman of the Section, Professor Haskell, presided at the opening of the meeting; the chairman-elect, Professor Allardice, then took the chair. The following officers were elected for the ensuing year: chairman, Professor Allardice; secretary, Dr. Buck; programme committee, Dr. Buck and Professors Blichfeldt and Manning.

It was voted that the time and place of the spring meeting be determined by the executive committee. The usual luncheon was served.

The following papers were presented at this meeting:

(1) Dr. B. A. BERNSTEIN: "A simplification of the Whitehead-Huntington set of postulates for Boolean algebras."

(2) Professor H. F. BLICHFELDT: "Some diophantine approximations."

(3) Dr. FRANK IRWIN: "A convergent series derived from the harmonic series."

(4) Dr. H. N. WRIGHT and Dr. FRANK IRWIN: "Some properties of the curves, $y =$ a polynomial in x."

(5) Mr. T. A. PIERCE: "The numerical factors of the arithmetic forms $\prod_{i=1}^{n}(1 \pm \alpha_i^m)$."

(6) Professor D. N. LEHMER: "Concerning certain binary forms of higher degrees than the second whose prime divisors are of the form $nx \pm 1$."

(7) Professor M. W. HASKELL: "A note on integration."

Dr. Wright was introduced by Dr. Irwin and Mr. Pierce by Professor Lehmer.

Abstracts of the papers follow.

1. The best set of postulates for Boolean logic, considered from the point of view of elegance and pedagogy, is that given by Whitehead and made rigorous by Huntington. The number of propositions in this set is ten and the number of special elements postulated three. Dr. Bernstein, by replacing three postulates of this set by a new proposition, while retaining elegance and naturalness, reduces the number of postulates to eight and the number of postulated special elements to one.

2. Let $X, Y, Z, \cdots$ be n linear homogeneous functions of n variables $x, y, z, \cdots$, having a determinant $\Delta \neq 0$. Minkowski has proved that such integers (not all zero) can be substituted for $x, y, z, \cdots$ that the sum $|X| + |Y| + |Z| + \cdots \leqq k\Delta^{1/n}$, where k is a function of n, independent of the coefficients involved in $X, Y, Z, \cdots$ (Geometrie der Zahlen, Leipzig, 1910). By applying the processes explained in "A new principle in the geometry of numbers," *Transactions*, 1914, Professor Blichfeldt obtains a much lower value for k, except for small values of n, than those given by Minkowski and by the author in the above article in the *Transactions*.

3. In extension of a result of Dr. Kempner in the *American Mathematical Monthly* for February, 1914, Dr. Irwin shows that we shall obtain a convergent series if we omit from the harmonic series all terms whose denominators contain the digits 9, 8, 7, $\cdots$ *each* more than a given number $a, b, c, \cdots$ of times respectively.

4. It is shown by Dr. Wright and Dr. Irwin that if to the curve $y = f(x)$, where $f(x)$ is a polynomial, all the tangents be drawn from a given point, then (1) the sum of their slopes and (2) the sum of the abscissas of their points of contact are independent of the ordinate of the given point. A remarkable line associated with the curve is noticed. Finally a graphical construction is given for the imaginary roots of a cubic equation and of certain biquadratics.

5. In volume 1 of the *American Journal of Mathematics* Lucas published a paper dealing with the properties of numbers given by $\dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ and $\alpha^n + \beta^n$, where α and β are the roots of

a quadratic equation having integral coefficients. Carmichael has given a different and more exhaustive treatment of these numbers in the *Annals of Mathematics*, 1913. In the present paper Mr. Pierce obtains somewhat similar results for numbers given by the forms $\prod_{i=1}^{n}(1 \pm \alpha_i^m)$, where the α_i denote algebraic integers defined as the roots of an nth degree equation. The forms of the factors of $\prod_{i=1}^{n}(1 - \alpha_i^m)$ are determined by use of algebraic number theory, and this perhaps constitutes the most novel result of the work.

6. Lucas has developed the theory of the prime divisors of the functions $U_n = (a^n - b^n)/(a - b)$ and $V_n = a^n + b^n$, where a and b are the roots of a quadratic equation (*American Journal of Mathematics*, volume 1, page 184). Connected with these functions are certain binary forms of degree equal to one half the totient of n, the divisors of which Professor Lehmer has shown to be of the form $2nx \pm 1$. Combining this result with certain results of Mr. Pierce, Professor Lehmer has also obtained a series of numbers the prime factors of which must belong to two such forms, thus restricting notably the character of their divisors.

7. Professor Haskell shows that the condition that a rational fraction whose denominator is the nth power of a quadratic should be rationally integrable, is that the numerator shall be of degree $2(n - 1)$ and that it shall be apolar to the $(n - 1)$st power of the quadratic factor of the denominator.

THOMAS BUCK,
Secretary of the Section.

TRANSFORMATION THEOREMS IN THE THEORY OF THE LINEAR VECTOR FUNCTION.

BY DR. VINCENT C. POOR.

(Read before the American Mathematical Society, December 31, 1915.)

SINCE the memorable work of Grassmann (1844), the study of the linear transformation has taken various forms, among which are the quaternions of Hamilton, the matrices of Cayley,

the dyadics of Gibbs, and the homography as treated by Burali-Forti and Marcolongo. The notation here followed is that of the book on "Transformations Linéaires" by the last mentioned authors. Reference to the French (1912) edition of this work will be briefly made by the letters B. M. with the section and number following.

The homography is defined by Burali-Forti and Marcolongo* as any linear operator which transforms vectors into vectors. One of the simplest examples of a homography is $\boldsymbol{u}\wedge$, the axial homography,† which transforms all vectors into vectors perpendicular to the vector $\boldsymbol{u}$.

Another concept of fundamental importance in advanced vector analysis is the Grassmann point derivative. If M and P are any two points of space, then $M - P$ is the vector represented by the line segment directed from P to M.‡ If we use $\boldsymbol{u}$ for the difference between any two points of space, and if $f(P)$ is a function depending on the point P, the differential of f, written df, δf, etc., may be defined briefly by the equation

$$df = \lim_{h \doteq 0} \frac{f(P + h\boldsymbol{u}) - f(P)}{h}.$$

From this it follows that

$$dP = \boldsymbol{u}$$

or dP is any arbitrary vector. The definition of the point derivative df/dP, following the Leibnitz notation, may now be expressed by the equation

$$\delta f = \frac{df}{dP}\delta P,\S$$

i. e., df/dP is an operator on any arbitrary vector δP which transforms that vector into δf. It may be shown that df/dP is a *linear* operator. In no sense is df/dP to be regarded as a quotient. When the operator df/dP transforms the operand, a vector, into a vector, the point derivative furnishes another example of a homography.

* B. M., 1.

† The cross $\times$ and the inverted $\vee$ ($\wedge$, read "vec"), standing between two vectors, indicate the scalar and vector products, respectively.

‡ The vector appears as a particular case of the Grassmann "first formation."

§ B. M., p. 60, 5.

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two arbitrary vectors, then the vector of a homography α, designated by $V\alpha$, may be defined as a vector such that

$$2V\alpha \times \boldsymbol{u} \wedge \boldsymbol{v} = \boldsymbol{v} \times \alpha\boldsymbol{u} - \boldsymbol{u} \times \alpha\boldsymbol{v}.\text{*}$$

It is easy to show that

$$\frac{d(\alpha\boldsymbol{u})}{dP}\boldsymbol{x} = \alpha\frac{d\boldsymbol{u}}{dP}\boldsymbol{x} + \left(\frac{d\alpha}{dP}\boldsymbol{x}\right)\boldsymbol{u},\dagger \tag{1}$$

where α is a homography and $\boldsymbol{u}$ and $\boldsymbol{x}$ are vectors. Transposing the first term of the right member, we have

$$\left(\frac{d\alpha}{dP}\boldsymbol{x}\right)\boldsymbol{u} = \left[\frac{d(\alpha\boldsymbol{u})}{dP} - \alpha\frac{d\boldsymbol{u}}{dP}\right]\boldsymbol{x}.$$

For the brackets in the right member, $S_P(\alpha, \boldsymbol{u})$ will be written. This binary operator is evidently a homography.

The rotational of a homography (written "Rot α"), may be briefly defined as a homography such that

$$(\text{Rot }\alpha)\boldsymbol{u} = 2VS(\alpha, \boldsymbol{u}). \tag{2}$$

The theorems of the present paper are linear transformation theorems which involve the homography, in general, as a function of two points of space. The letter P will be used throughout as the point of integration. It will be understood that the surface σ bounds the region τ and that $\boldsymbol{n}$ is a unit normal at a point P of the surface σ with its positive sense towards the interior of σ.

THEOREM 1. *If α is a homography symmetric in P and M, such that*

$$\frac{d\alpha}{dM} = -\frac{d\alpha}{dP},$$

and if $\boldsymbol{u}$ is independent of M, then

$$2\int V\frac{d(\alpha\boldsymbol{u})}{dM}d\tau = \int \boldsymbol{n} \wedge \alpha\boldsymbol{u}d\sigma + 2\int V\left(\alpha\frac{d\boldsymbol{u}}{dP}\right)d\tau.$$

For, taking $\boldsymbol{x}$ as a function of M alone, and applying (1) and the hypotheses of the theorem, we find that

$$\frac{d(\alpha\boldsymbol{u})}{dM}\boldsymbol{x} = \left(\frac{d\alpha}{dM}\boldsymbol{x}\right)\boldsymbol{u} = -\left(\frac{d(\alpha\boldsymbol{u})}{dP} - \alpha\frac{d\boldsymbol{u}}{dP}\right)\boldsymbol{x}.$$

* B. M., 8 [2].
† B. M., 36 [1].

Remembering the definition of S this result may be written

$$\frac{d(\alpha \boldsymbol{u})}{dM} = - S_P(\alpha, \boldsymbol{u})$$

or by using (2), the definition for the rotational of a homography, we have

$$2V\frac{d(\alpha \boldsymbol{u})}{dM} = - 2VS_P(\alpha, \boldsymbol{u}) = - (\mathrm{Rot}_P\, \alpha)\boldsymbol{u}.$$

By substituting this result in the known transformation formula

$$\int (\mathrm{Rot}_P\, \alpha)\boldsymbol{u} d\tau = - \int \boldsymbol{n} \wedge \alpha \boldsymbol{u} d\sigma - 2\int V\left(\alpha \frac{d\boldsymbol{u}}{dP}\right) d\tau,^*$$

the theorem will be obtained. This theorem may be put into a slightly different form if we introduce the definition for the rotational† of a vector, written "rot," which is expressed by the equation

$$\mathrm{rot}_P\, \boldsymbol{u} = 2V\frac{d\boldsymbol{u}}{dP}.$$

The theorem will then read

$$\int \mathrm{rot}_M \alpha \boldsymbol{u} d\tau = \int \boldsymbol{n} \wedge \alpha \boldsymbol{u} d\sigma + 2\int V\left(\alpha \frac{d\boldsymbol{u}}{dP}\right) d\tau.$$

In expressing the next theorem we will need the conjugate of a homography $K\alpha$, the first invariant of a homography $I_1\alpha$, and the gradient of a homography grad α, of which the gradient of a scalar is a special case. The Maxwell divergence of a vector, which arises, is to be found in any book on the elements of vector analysis. The conjugate of a homography may be defined by the equation

$$K\alpha = \alpha - 2V\alpha \wedge,$$

which is again a homography. The first invariant of a homography α is a scalar, which, for any three arbitrary vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, satisfies the following relation:

$$\boldsymbol{u} \wedge \boldsymbol{v} \times \boldsymbol{w} \cdot I_1\alpha = \boldsymbol{v} \wedge \boldsymbol{w} \times \alpha\boldsymbol{u} + \boldsymbol{w} \wedge \boldsymbol{u} \times \alpha\boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v} \times \alpha\boldsymbol{w}.$$

* B. M., 56 [3], second.

† The rotational of a vector is identical with the Maxwell curl, or the Gibbs $\nabla \times$.

The gradient of a homography α, function of the point P, is a vector such that, for any arbitrary vector $\boldsymbol{u}$,

$$\operatorname{grad}_P \alpha \times \boldsymbol{u} = I_1 S_P(K\alpha, \boldsymbol{u}).*$$

THEOREM 2. *If α is a homography function of P and M, such that*

$$\frac{d\alpha}{dM} = -\frac{d\alpha}{dP},$$

and if $\boldsymbol{u}$ is a function independent of M, then

$$\int \operatorname{div}_M \alpha \boldsymbol{u} d\tau = \int \boldsymbol{n} \times \alpha \boldsymbol{u} d\sigma + \int I_1 \left(\frac{d\boldsymbol{u}}{dP} \alpha \right) d\tau.$$

The proof of this theorem follows easily from the following known theorems, namely:

$$\operatorname{div}_M \alpha \boldsymbol{u} = \boldsymbol{u} \times \operatorname{grad}_M K\alpha + I_1 \left(\alpha \frac{d\boldsymbol{u}}{dM} \right),\dagger$$

which reduces, since $\boldsymbol{u}$ is independent of M, to

$$\operatorname{div}_M \alpha \boldsymbol{u} = -\boldsymbol{u} \times \operatorname{grad}_P K\alpha,$$

and

$$\int \operatorname{grad}_P \alpha \times \boldsymbol{u} d\tau = -\int \boldsymbol{u} \times \alpha \boldsymbol{n} d\sigma - \int I_1 \left(\frac{d\boldsymbol{u}}{dP} K\alpha \right) d\tau,\ddagger$$

which may be written

$$\int \boldsymbol{u} \times \operatorname{grad}_P K\alpha d\tau = -\int \boldsymbol{u} \times K\alpha \boldsymbol{n} d\sigma - \int I_1 \left(\frac{d\boldsymbol{u}}{dP} \alpha \right) d\tau.$$

The substitution of $-\operatorname{div}_M \boldsymbol{u}$ for its equal in the left member of the last equation and the application of the commutation theorem

$$\boldsymbol{n} \times \alpha \boldsymbol{u} = \boldsymbol{u} \times K\alpha \boldsymbol{n}$$

lead to the theorem as stated.

The dyad $H(\boldsymbol{u}, \boldsymbol{v})$, another binary operator, is defined by the equation

$$H(\boldsymbol{u}, \boldsymbol{v}) x = \boldsymbol{u} \times \boldsymbol{x} \cdot \boldsymbol{v}.$$

The dyad is thus seen to be a homography. We may now express

---

* B. M., 8 and 37 [3].

† B. M., 41 [3], second.

‡ B. M., 56 [2], first.

THEOREM 3. *If α is a homography, symmetric in P and M, such that*

$$\frac{d\alpha}{dM} = -\frac{d\alpha}{dP},$$

and if $\boldsymbol{u}$ is independent of M, then

$$\int \frac{d(\alpha \boldsymbol{u})}{dM} d\tau = \int H(\boldsymbol{n}, \alpha \boldsymbol{u}) d\sigma + \int \alpha \frac{d\boldsymbol{u}}{dP} d\tau.$$

To demonstrate this theorem we may first observe that

$$\int (\operatorname{div}_P \boldsymbol{v}) \cdot \boldsymbol{u} d\tau = -\int \boldsymbol{v} \times \boldsymbol{n} \cdot \boldsymbol{u} d\sigma - \int \frac{d\boldsymbol{u}}{dP} \boldsymbol{v} d\tau.*$$

Replacing $\boldsymbol{v}$ by $\boldsymbol{x}$ and $\boldsymbol{u}$ by the vector $\alpha \boldsymbol{u}$, this formula will become

$$\int (\operatorname{div}_P \boldsymbol{x}) \cdot \alpha \boldsymbol{u} d\tau = -\int \boldsymbol{n} \times \boldsymbol{x} \cdot \alpha \boldsymbol{u} d\sigma - \int \frac{d(\alpha \boldsymbol{u})}{dP} d\tau.$$

Under the assumption that $\boldsymbol{x}$ is independent of P this readily reduces to

$$\int S_P(\alpha, \boldsymbol{u}) d\tau = -\int H(\boldsymbol{n}, \alpha \boldsymbol{u}) d\sigma - \int \alpha \frac{d\boldsymbol{u}}{dP} d\tau.$$

With the same restriction on the $\boldsymbol{x}$ we have

$$\frac{d(\alpha \boldsymbol{u})}{dM} \boldsymbol{x} = \left(\frac{d\alpha}{dM} \boldsymbol{x}\right) \boldsymbol{u} = -S_P(\alpha, \boldsymbol{u}) x.$$

From these considerations the theorem will be evident. When the surface integral is identically zero, we have the useful corollary

$$\int \frac{d(\alpha \boldsymbol{u})}{dM} d\tau = \int \alpha \frac{d\boldsymbol{u}}{dP} d\tau.$$

This situation could well happen in the case of an infinite region.

THEOREM 4. *If α is a homography function of P and M and if $\boldsymbol{u}$ is independent of M, and $\boldsymbol{x}$ independent of P, then*

$$\int \frac{d\left(\alpha \frac{d\boldsymbol{u}}{dP}\right)}{dM} \boldsymbol{x} d\tau = \int \left(\frac{d\alpha}{dM} \boldsymbol{x}\right) \frac{d\boldsymbol{u}}{dP} d\tau.$$

---

* B. M., 56 [3], first.

This theorem is demonstrated as soon as the integrands are shown to be equal. Introducing the vector $\boldsymbol{y}$, a function of M alone, then the expression

$$\frac{d\left(\alpha \dfrac{d\boldsymbol{u}}{dP}\boldsymbol{y}\right)}{dM}\boldsymbol{x}$$

may be written in two different forms,* namely

$$\alpha \frac{d\left(\dfrac{d\boldsymbol{u}}{dP}\boldsymbol{y}\right)}{dM}\boldsymbol{x} + \left(\frac{d\alpha}{dM}\boldsymbol{x}\right)\frac{d\boldsymbol{u}}{dP}\boldsymbol{y}$$

and

$$\alpha \frac{d\boldsymbol{u}}{dP}\frac{d\boldsymbol{y}}{dM}\boldsymbol{x} + \frac{d\left(\alpha \dfrac{d\boldsymbol{u}}{dP}\right)}{dM}\boldsymbol{y}.$$

But by the hypotheses of the theorem, the first terms of these expressions may be seen to be equal. The last two are then equal and the theorem follows.

THEOREM 5. *If α is a homography symmetric in P and M and if $\boldsymbol{u}$ is a function independent of M, then*

$$\int \Delta_M'(\alpha\boldsymbol{u})d\tau = -\int \left\{\left(\frac{d\alpha}{dP}\boldsymbol{n}\right)\boldsymbol{u} - \left(\alpha\frac{d\boldsymbol{u}}{dP}\right)\boldsymbol{n}\right\} d\sigma + \int \alpha(\Delta_P'\boldsymbol{u})d\tau.$$

The new symbols involved in this theorem may be defined as follows:

$$\Delta_P'\boldsymbol{u} = \operatorname{grad}\frac{d\boldsymbol{u}}{dP},$$

$$(\Delta_P\alpha)\boldsymbol{u} = \operatorname{grad}\left\{\frac{d(\alpha\boldsymbol{u})}{dP} - 2\alpha\frac{d\boldsymbol{u}}{dP}\right\} + \alpha \operatorname{grad}\frac{d\boldsymbol{u}}{dP}.$$

From these definitions it follows at once that

$$\Delta_P'(\alpha\boldsymbol{u}) = (\Delta_P\alpha)\boldsymbol{u} - \alpha(\Delta_P'\boldsymbol{u}) + 2\operatorname{grad}_P\left(\alpha\frac{d\boldsymbol{u}}{dP}\right).\dagger$$

---

* Apply equation (1).

† B. M., 50 [3]. The parentheses may be omitted, since Δ operates on homographies only, while Δ' operates only on vectors.

Applying our hypotheses to this equation we have

$$\Delta'_M(\alpha \boldsymbol{u}) = (\Delta_P \alpha)\boldsymbol{u}.$$

This theorem, then, becomes the direct consequence of the formula

$$\int (\Delta_P \alpha)\boldsymbol{u} d\tau = -\int \left\{ \left(\frac{d\alpha}{dP}\boldsymbol{n}\right)\boldsymbol{u} - \left(\alpha\frac{d\boldsymbol{u}}{dP}\right)\boldsymbol{n} \right\} d\sigma + \int \alpha(\Delta'\boldsymbol{u})d\tau.*$$

Another form of this theorem is

$$\int \Delta'_M(\alpha \boldsymbol{u})d\tau = \int (\Delta_P \alpha)\boldsymbol{u} d\tau,$$

which holds even if the subscript P be replaced by M provided, of course, that $\boldsymbol{u}$ is restricted by the conditions of the theorem.

THEOREM 6. *If* $\alpha \cdot d\beta = d\beta \cdot \alpha$, *then*

$$\int \alpha \text{ grad } \beta d\tau = -\int \alpha\beta\boldsymbol{n} d\sigma - \int \beta \text{ grad } \alpha d\tau.$$

This theorem is easily proved by using the addition theorem for grad $\alpha\beta$ and by applying the "gradient theorem"†

$$\int \text{grad } \alpha d\tau = -\int \alpha\boldsymbol{n} d\sigma.$$

As a special consequence of this one finds, upon replacing β by the scalar m, that

$$\int \alpha \text{ grad } m d\tau = -\int m\alpha\boldsymbol{n} d\sigma - \int m \text{ grad } \alpha d\tau.$$

The following associated theorem may be proved in a manner similar to that just suggested:

THEOREM 7. *If* $\alpha \cdot d\beta = d\beta \cdot \alpha$ *then*

$$\int (\text{Rot } \alpha)\beta d\tau = -\int \boldsymbol{n} \wedge \alpha\beta d\sigma - \int (\text{Rot } \beta)\alpha d\tau.$$

The purpose of the first five theorems is to transfer the derivative operator from the formal product $\alpha\boldsymbol{u}$ to the homography α or to the vector $\boldsymbol{u}$ alone. The need for such transformations arises naturally in certain studies in applied mathematics.

THE UNIVERSITY OF MICHIGAN,
October 6, 1915.

---

* B. M., 56 [3], third.
† B. M., 55 [3].

NAPIER'S DESCRIPTIO AND CONSTRUCTIO.

John Napier and the Invention of Logarithms, 1614. A lecture by E. W. Hobson. Cambridge, University Press, 1914. 48 pp. Cloth, price 1s. 6d.

Napier and the Invention of Logarithms. By G. A. Gibson. Glasgow, 1914.* 24 pp. Paper.

Some reference has been already made in this Bulletin† to the celebration in 1914, under the auspices of the Royal Society of Edinburgh, of the tercentenary of one of the greatest events in the history of science, namely the publication of John Napier's Mirifici Logarithmorum Canonis Descriptio. According to Professor Hobson this work "embodies one of the very greatest scientific discoveries that the world has seen." J. W. L. Glaisher's judgment, which seems to be also that of Professor Gibson (page 13), is, that "with the exception of the Principia of Newton, there is no mathematical work published in the country which has produced such important consequences, or to which so much interest attaches as to Napier's Descriptio."‡

Nor was marked enthusiasm with regard to this work lacking among Napier's contemporaries of eminence. This was especially true of Kepler and Henry Briggs (1556–1630), professor of mathematics in London. In his Ephemeris for 1620 Kepler published as the dedication a letter addressed to Napier, congratulating him warmly on his invention and on the benefit he had conferred upon astronomy. Kepler explains how he verified the canon and found no essential errors in it, beyond a few inaccuracies near the beginning of the quadrant. The letter was written July 28, 1619, about two years after Napier's death, of which Kepler had not heard.

And as to Briggs, he wrote thus to Archbishop Ussher in the year 1615:§ "Napper, lord of Markinston, hath set my

* This pamphlet is a reprint, with separate title-page, of an article in the *Proceedings of the Royal Philosophical Society in Glasgow*. This article is also reprinted in the very interesting and useful volume: Modern Instruments and Methods of Calculation. A Handbook of the Napier Tercentenary Exhibition, edited by E. M. Horsburgh, London [1914], pp. 1–16. For a review of this work by Professor D. E. Smith, see *Science*, n. s., vol. 42, July 23, 1915, pp. 128–129.

† Vol. 21, Dec., 1914, pp. 123–127.

‡ Article "Logarithms" in Encyclopædia Britannica, 11th edition, 1911.

§ Ussher's Letters, 1686, p. 36.

head and hands at work with his new and admireable Logarithms. I hope to see him this summer, if it please God, for I never saw a book which pleased me better and made me more wonder."* Again, Thomas Smith in his Life of Briggs† says of him (Gibson, page 16) when describing his enthusiasm over the Canon Mirificus: "He cherished it as the apple of his eye; it was ever in his bosom or in his hand, or pressed to his heart, and, with greedy eyes and mind absorbed, he read it again and again. . . . It was the theme of his praise in familiar conversation with his friends, and he expounded it to his students in the lecture room."

Briggs paid his anticipated visit to Napier in the summer of 1615, and William Lilly, the astrologer, has told us‡ of his reception at Merchiston: "I will acquaint you with one memorable story, related unto me by John Marr, an excellent mathematician and geometrician, whom I conceive you remember. He was servant to King James I and Charles I. When Merchiston first published his Logarithms Mr. Briggs,

* Sir John Leslie (1766–1832), for many years professor of mathematics and physics in Edinburgh University, was less restrained in his admiration, and characterizes the invention of logarithms as "the noblest conquest ever achieved by man" (Philosophy of Arithmetic, Edinburgh, 1817, p. 156). In a few copies of the English edition of the Constructio published at London in 1616, a cancelled leaf with eulogistic Spencerian verses by one Thomas Bretnor, has not been cut out. These verses begin:

> Pull off your laurel rayes, you learned Greekes
> Let Archimed and Euclid both give way,
> . . . .

and the final stanzas are:

> And bonnets vaile; you Germans! Rheticus,
> Reignoldus, Oswald and John Regiomont,
> Lansbergius, Finckius and Copernicus,
> And thou Pitiscus, from whose clearer font
> We sucked have the sweet from Hellespont.
> For were your labours ne'er composed so well
> Great Napier's worth they could not parallel.
>
> By thee great Lord we solve a tedious toyle,
> In resolution of our trinall lines,
> We need not now to carke, to care, or moile,
> Sith from thy witty braine such splendour shines,
> As dazels much the eyes of deepe divines.
> Great the invention, greater is the praise,
> Which thou unto thy nation hence dost raise."

† Vitae quorundam eruditissimorum et illustrium virorum. Londini, 1707.

‡ Lilly's Life, London, 1721.

then reader of the astronomy lectures at Gresham College in London, was so surprised with admiration of them that he could have no quietness in himself until he had seen that noble person whose only invention they were. He acquaints John Marr therewith, who went into Scotland before Mr. Briggs, purposely to be there when these two so learned persons should meet. Mr. Briggs appoints a certain day when to meet at Edinburgh; but, failing thereof, Merchiston was fearful he would not come. It happened one day as John Marr and the Lord Napier were speaking of Mr. Briggs, 'Oh! John,' saith Merchiston, 'Mr. Briggs will not come now'; at the very instant one knocks at the gate, John Marr hastened down and it proved to be Mr. Briggs to his great contentment. He brings Mr. Briggs into my Lord's chamber, where almost one quarter of an hour was spent, each beholding other with admiration, before one word was spoken. At last Mr. Briggs began,—'My Lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto astronomy, *viz.* the Logarithms; but, my Lord, being by you found out, I wonder nobody else found it out before, when now being known it appears so easy."

Napier and Briggs must have been kindred spirits, for Briggs spent a month at Merchiston, and returned the following summer. He had also planned another visit in the summer of 1617, but Napier's death intervened.

The Descriptio is a small quarto containing an ornamental title page,* preface, 57 pages of explanatory matter, and 90 pages of tables, one page of which is reproduced by Hobson. The preface contains Napier's own account of his invention: "Seeing there is nothing (right well-beloved students of the Mathematics) that is so troublesome to mathematical practice; nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances. And having thought

* A facsimile is given in Mark Napier's Memoirs of John Napier of Merchiston, his lineage, life and times, with a history of the invention of logarithms. London, 1834, p. 374.

upon many things to this purpose, I found at length some excellent brief rules to be treated of (perhaps) hereafter. But amongst all, none more profitable than this which together with the hard and tedious multiplications, divisions and extractions of roots, doth also cast away from the work itself even the very numbers themselves that are to be multiplied, divided and resolved into roots, and putteth other numbers in their place which perform as much as they can do, only by addition and subtraction, division by two or division by three."

The explanatory matter contains "an account of Napier's conception of a logarithm, of the principal properties of logarithms and also of their application in the solution of plane and spherical triangles." Napier's well-known rules of circular parts* containing the complete system of formulas for the solution of right-angled spherical triangles are here given. The tables include the logarithms of the sines of angles from 0° to 90° at intervals of one minute, to seven or eight places of decimals. The arrangement is semiquadrantal, so that the "differentiæ" which are the differences of logarithms in the same line, are the logarithms of the tangents. It should be borne in mind that, "at that time and long afterwards, the sine of an angle was not regarded, as at present, as a ratio but as a length of that semi-chord of a circle of given radius which subtends the angle at the centre. Napier took the radius to consist of 10^7 units, and thus the sine of 90°, called the whole sine, is 10^7; the sines of smaller angles decreasing to zero. The table is therefore one of the logarithms of numbers between 10^7 and 0, not for equidistant numbers, but for the numbers corresponding to equidistant angles." Napier's logarithms are *not* the logarithms now termed Napierian or hyperbolic, that is to say, logarithms to the base e where $e = 2.718\ldots$; the relation between N (a sine) and L its logarithm, as defined in the Descriptio, being $N = 10^7 e^{-L/10^7}$, so that (ignoring the factors 10^7, the effect of which is to render sines and logarithms integral to seven figures), the base is $1/e$. If l (says Glaisher, l. c.) denotes the logarithm to the base e (that is, the so-called "Napierian" or hyperbolic logarithm) and L denotes, as above, Napier's logarithm, the connection between l and L is expressed by

* Cf. I. Todhunter's Spherical Trigonometry revised by Leathem, London, 1901, p. 51 ff. for a discussion of Napier's proof. Reference may also be given to a note by E. O. Lovett in this BULLETIN, vol. 4, July, 1898, pp. 552–554.

$$L = 10^7 \log_e 10^7 - 10^7 l, \quad \text{or} \quad e^l = 10^7 e^{-L/10^7}.$$

In an "admonitio" on the last page Napier states that he will publish the mode of construction of the canon in the Descriptio "si huius inventi usum eruditio gratum fore intellexero." Napier did not live to keep this promise. The proposed work had been written however and was published by his son, Robert Napier (assisted by Henry Briggs), in 1619, under the title: Mirifici logarithmorum canonis constructio. It consists of two pages of preface and 67 pages of text. In the text we find a full account of the method of construction of the canon. There is also here, expressed in words, one of the four formulas for the solution of spherical triangles, known as Napier's analogies; the other three, easily deduced from Napier's result, were formulated by Briggs in the appendix. In the Constructio logarithms are called "artificial numbers" (numeri arteficiales) and Robert Napier states that the work was composed several years before Napier had invented the name logarithm.* The Constructio may therefore have been written many years before the publication of the Descriptio in 1614.

The decimal point in arithmetic appears to have been independently invented by Napier. Decimal fractions were first introduced by Stevin in a tract entitled De Thiende and published in 1585; but his notation is excessively cumbrous. Stevin would have written 652(0)1(1)3(2)7(3)† or $\overset{0}{652}\overset{1}{1}\overset{2}{3}\overset{3}{7}$, instead of Napier's notation 652·137. It was not till much later, however, that the equivalent of Napier's notation was generally used.

---

* This term is derived from two Greek words meaning "the number of the ratios." For explanation of the appropriateness of the term consult H. S. Carslaw's paper "The discovery of logarithms by Napier," *Math. Gazette*, vol. 8, p. 82. (The whole paper is given pp. 76–84; 115–119, May, July, 1915. Another paper with the same title was published by Professor Carslaw in *Jour. and Proc. Roy. Soc. New South Wales*, vol. 48, 1914, pp. 42–72.)

It did not take long for the word logarithm to come into English literature. In Act I, scene 1 of Ben Jonson's comedy, "The Magnetic Lady," which was performed in 1632 and first published in 1640, the following lines occur:

> "Sir Interest . . . .
> . . . will tell you instantly, by Logarythmes,
> The utmost profit of a stock imployed;"

† A facsimile of a page with this notation from the French edition of Stevin's work is given by D. E. Smith in "History of Decimal Fractions," *Teachers College Bulletin*, (1), no. 5 (March 12, 1910).

There have been several different editions of the Descriptio and Constructio. Of the former in Latin, there were issues in 1614, 1619, 1620, 1658, 1807,* 1857 and 1899; there were English editions in 1616, 1618, 1857. The Latin editions of the latter were in 1619, 1620, 1658 and 1899;† an English edition published by W. R. Macdonald‡ at Edinburgh in 1889 is a scholarly work and contains a professedly complete catalogue of various editions of Napier's works. The bibliophile may find titles supplementary to this catalogue in Quaritch's catalogue no. 336 (March, 1915); good copies of the first English or Latin editions of the Descriptio and of the first Latin edition of the Constructio are worth about $100 apiece.

It is not within the scope of the title which I have chosen for this paper to do more than point out that: (1) it was wholly independent of suggestion from any one else that Napier recognized the advantages, and indicated means for calculation, of a system of logarithms in which $\log 1 = 0$ and $\log 10 = 10^{10}$. "This is practically equivalent to the assumption $\log 10 = 1$, as the former assumption merely indicates that the logarithms are to be calculated to 10 places of decimals." (2) The system of logarithms invented by Joost Bürgi (1552–1632), a Swiss watchmaker and instrument-maker, is decidedly inferior to that of Napier, "and the knowledge of the use of logarithms which was spread in the scientific world was entirely due to Napier."

Both of the sketches under review are remarkably interesting and should be in every mathematical library. Professor Hobson develops, at some length, Napier's methods in connection with his tables.§ This is only briefly touched upon by Professor Gibson who sets forth, in his wonted attractive style, the main facts of Napier's life, dwelling especially upon his intimate relations with Briggs. This latter sketch is the longer although it contains the smaller number of pages.

R. C. ARCHIBALD.

BROWN UNIVERSITY,
PROVIDENCE, R. I.

* F. Maseres, Scriptores logarithmici. London, vol. 6, 1807.

† Both the Descriptio and Constructio are reprinted by N. W. L. A. Gravelaar in *Verhandelingen der Kon. Akad. van Wet. te Amsterdam*, 1 sectie, deel 6, 1899.

‡ Mr. Macdonald is also the author of the life of Napier in the Dictionary of National Biography.

§ Somewhat extended analysis of Napier's logarithmic works is given by Delambre in his Histoire de l'Astronomie moderne, tome 1, Paris, 1821, pp. 491–506. See also A. von Braunmühl's Vorlesungen über Geschichte der Trigonometrie, zweiter Teil. Leipzig, 1903.

MATHEMATICAL QUOTATION BOOKS.

Memorabilia Mathematica or the Philomath's Quotation-Book. By ROBERT EDOUARD MORITZ. New York, Macmillan, 1914. xiv + 410 pp. Price $3.00.

THE United States has now joined France and Germany in contributing to this class of literature.

It is some twenty-five years since A. M. Rebière published a pioneer work entitled: Mathématiques et mathématiciens. Pensées et curiosités.* The second edition (in later editions there is practically no change) was divided into five parts, headed: Morceaux choisis et pensées (pages 1–178), Variétés et anecdotes (179–340), Paradoxes et singularités (341–470), Problèmes curieux et humoristiques (471–526), Note bibliographique, index and table des matières (527–566).

The textual parts contain, mainly, short quotations or translations (the whole book is in French) from writings ancient and modern. When there is any indication of the source nothing is given, except in rare cases, but the name of the author; the composition of the numerous unsigned paragraphs is attributable to the editor. The work is not, then, strictly a book of quotations but a sort of admixture of quotations, history, mathematical recreations, and table-talk. In defense of his mixture of things gay and serious the author makes appeal to the authority of Pascal: "Les matières de géométrie sont sérieuses d'elles-mêmes, qu'il est avantageux qu'il s'offre quelque occasion pour les rendre un peu divertissantes."

Ahrens's Scherz und Ernst in der Mathematik† differs essentially from the work of Rebière. In the first place the former is strictly a book of quotations; secondly, each quotation is invariably given in the original language, spoken or written; thirdly, exact bibliographical data are provided for all quotations; fourthly, the quotations follow one another consecutively from pages 1 to 495 without grouping under subject headings. A 24-page detailed index of subjects and authors provides the means for rapid orientation. Names of living mathematicians are rarely met with, but references to

* Deuxieme édition, revue et augmentée, Paris, 1893. 2+566 pp.; 4e éd., Paris, 1911.

† W. Ahrens, Scherz und Ernst in der Mathematik. Geflügelte und ungeflügelte Worte. Leipzig, 1904.

the "old masters" such as Abel, Euclid, Euler, Gauss, Helmholtz, Lagrange, Laplace, Steiner, and Weierstrass, are very numerous.

The whole constitutes a most admirable piece of work and must long serve as a desirable model for works of like nature.

The book under review is also a quotation book. The author tells us that ten years were devoted to its preparation and that as a result there have been brought together some 1,200 "more or less familiar passages* pertaining to mathematics, by poets, philosophers, historians, statesmen, scientists, and mathematicians. These have been gathered from over three hundred authors and have been grouped under twenty heads and cross indexed under nearly seven hundred topics." The headings are: Definition and object of mathematics; The nature of mathematics; Estimates of mathematics; The value of mathematics; The teaching of mathematics; Study and research in mathematics; Modern mathematics; The mathematician; Persons and anecdotes; Mathematics as a fine art, as a language; Mathematics and logic, and philosophy, and science; Arithmetic; Algebra; Geometry; The calculus and allied topics; The fundamental concepts of time and space; Paradoxes and curiosities.

Apart from classification in this way the method of Rebière is followed by giving a translation of all quotations in a foreign language, and in probably not more than a score of cases is the original also given. This departure from the scheme of Ahrens must be considered as a great defect, when the book is employed as a work of reference. No scholar is likely to use the translation of a quotation unless he can test its faithfulness by comparison with the original. It is indeed true that exact reference is usually given for this, but then some of the desirable usefulness of the single volume has disappeared.

The compiler has avoided as far as possible traversing the ground that has been trodden already by Rebière and Ahrens. "Thus certain topics, as the correspondence of German and French mathematicians, so excellently treated by Ahrens, have purposely been omitted. The repetitions are limited to a small number of famous utterances whose absence from a work of this kind could scarcely be defended on any grounds."

---

* This total is about the same as in Ahrens's work. It is no doubt due to the special method of numbering the quotations that some reviewers estimated the total to exceed 2,000; cf. *Mathematical Gazette*, vol. 8 (March, 1915), p. 57; *Nature*, vol. 94 (Oct. 8, 1914), p. 144.

The number of quotations from contemporary or living mathematicians is large. Appended to numerous quotations from the writings of Americans, the names of Bôcher, Cajori, Emerson, Keyser, Myers, B. Peirce, D. E. Smith, W. F. White, E. B. Wilson, J. W. Young, and J. W. A. Young, each of which occurs at least three times, may be noted. John Wesley Young is referred to as *Charles* Wesley Young on page 6. To De Morgan 50 quotations are attributed; to Sylvester 38, to Klein 20, to Shakespeare 3, and so on. As recent biographies have been drawn upon for anecdotes, Sylvanus P. Thompson's Life of Lord Kelvin has not been overlooked. It may be of interest to make a single quotation. First note that elsewhere in this BULLETIN,* I have called attention to the integral $\int_0^\infty e^{-x^2}dx$ which plays an important rôle in the theory of errors, to its connection with certain other integrals studied by Euler and with the curve named by Cesàro the clothoïde, which was considered by Jacob Bernoulli, and by Fresnel in discussing the diffraction of light. The quoted footnote near the end of Thompson's biography is as follows:

"Once when lecturing to a class he [Lord Kelvin] used the word 'mathematician,' and then interrupting himself asked his class: 'Do you know what a mathematician is?' Stepping to the blackboard he wrote upon it:

$$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}.$$

Then putting his finger on what he had written, he turned to his class and said: 'A mathematician is one to whom *that* is as obvious as that twice two makes four is to you. Liouville was a mathematician.' Then he resumed his lecture."

While Mr. Moritz's work seems to have been very carefully compiled, it is by no means free from at least minor errors (or misprints) and misrepresentations. For example in quotation no. 1043, Mr. Macfarlane is said to have written: "Maxwell denoted Thomson by T and Tait by T': so that it became customary to quote Thomson and Tait's Treatise on Natural Philosophy as T and T'." Instead of T' should of course be T'. In his sketch of Tait, J. S. MacKay refers to "*T and T′* (Thomson et Tait), la notation prolongée T''

* Vol. 20 (June, 1914), pp. 488–489.

servant, chez les amis de Tait, à designer le professeur Tyndall."*

In no. 1049 we read: "His only reply was that he could impossibly interrupt his work"; in 1858 the name of the translator of Dante should be given as Cary; the original of A. C. Orr's mnemonic for π has "In rhymes unapt," not "inapt"; on page 405 of the index for Reid, M. read Reid, T.; all the references after Pope on page 404 are wrong; and on page 410 two quotations are incorrectly attributed to J. W. A. Young instead of J. W. Young.

To state (no. 1007) that Alexander Pope's "Epitaph intended for Sir Isaac Newton" was:

> Nature and Nature's laws lay hid in night:
> God said, "Let Newton be!" and all was light.

is inaccurate. It would have been an easy matter to have turned to the standard edition of Pope's works and found† that the intended epitaph for Westminster Abbey was as follows:

> ISAACUS NEWTONUS
> QUEM IMMORTALEM
> TESTANTUR TEMPUS, NATURA, COELUM:
> MORTALEM
> HOC MARMOR FATETUR
> Nature and Nature's laws lay hid in night:
> God said, *Let Newton be!* and all was light.

Again J. Spence and James Porton are given as authorities for statements (nos. 1023 and 1024) by Newton. Had Sir David Brewster's standard Life of Newton been consulted‡ it would have been found that the first line of 1023 was inaccurate while no. 1024 is made to convey an entirely wrong idea. Instead of "I don't know what I may seem to the world" of the former should be, "I do not know what I may appear to the world." No. 1024 is given as: "If I have seen farther than Descartes, it is by standing on the shoulders of giants." In the course of a letter addressed to Robert Hooke and dated "Cambridge, February 5, 1675–6," occur the sentences: "But, in the mean time, you defer too much to my ability in searching into this subject. What Descartes did was a good step. You have added much several ways, and especially in considering the colours of thin plates. *If I have seen farther, it is by standing on the shoulders of giants.*"

---

* *L'Enseignement mathématique*, vol. 7, 1905, p. 7.

† Works, edited by Elwin and Courthope, vol. 4, London, 1882, p. 300.

‡ Memoirs of the Life, Writings and Discoveries of Sir Isaac Newton. 2 vols. Edinburgh, 1855, vol. 1, p. 142; vol. 2, p. 407.

But this selection from examples must suffice to illustrate the criticisms made above with regard to errors and careless presentation.

As to additions, I suggest two or three, at random, in connection with things geometrical.

Why not give a reference for Hamilton's "letter to De Morgan (1852)" with regard to the construction of the regular polygon of 17 sides?* And would not the reproduction of Gauss's original announcement of the discovery of the possibility of construction of such a polygon, with ruler and compass,† be worth while?

Why leave out Prior's

"Circles to square, and cubes to double,
Would give a man excessive trouble;"‡

And finally, might not the plan of the work permit the inclusion of the verses of the British Museum MS. which shows that Euclid was studied in England as far back as 924–940 A. D.?

The clerk Euclyde on this wyse hit fonde
Thys craft of gemetry yn Egypte londe
Yn Egypte he tawghte hut ful wyde,
Yn dyvers londe on every syde.
Mony erys afterwarde y vnderstonde
Gher that the craft com ynto thys londe.
Thys craft com ynto England, as y ghow say,
Yn tyme of good kyng Adelston's day.§

R. C. Archibald.

Brown University,
Providence, R. I.

SHORTER NOTICES.

The Development of Arabic Numerals in Europe. By G. F. Hill. Oxford, Clarendon Press, 1915. 125 pp. Price 7 shillings 6 pence.

It is a commonplace remark that noteworthy achievements in this world often have their inception in the most trivial incidents, and this semiparadoxical law is well illustrated in the work under review. Mr. Hill is the curator of the depart-

* Graves's Life of Sir Wm. R. Hamilton, vol. 3 (1889), pp. 433–435.

† *Intelligenzblatt der allgem. Literatur-Zeitung*, Nr. 66, 1 Junius, 1796, col. 554.

‡ In Alma, canto 3, lines 366–7, published in 1717. Or in Poetical Works of Matthew Prior in 2 volumes, London, 1779, vol. 1, p. 404.

§ J. O. Halliwell, Rara Mathematica, second ed., London, 1841, p. 56.

ment of coins and medals in the British Museum, and a few years ago the date (1481) of an Italian medal of the Sultan Mahomet II, by Costanzo of Ferrara, being called in question, he set about the study of the forms of the so-called Arabic numerals of the early Renaissance period. This naturally led him to extend his researches back to the period of probable introduction of these numerals in Europe, and forward to the time when, owing chiefly to the influence of printing, the forms became practically fixed. It is needless to say that Mr. Hill's search soon carried him beyond the field of numismatics and sigillography, and into the general domain of epigraphy, of medieval manuscripts, and of early typography, so that his essay becomes of value not merely in the field in which he is one of the great living authorities, but in other fields as well.

Such a study has long been needed to assist students in the history of mathematics; for the problem of the date of the introduction of our present numerals into Europe is by no means solved, and aids of this kind are exceedingly valuable. For example, the dates of 800 on the sarcophagus of Pegavus Petrasanta in Milan, of 1007 on a gravestone at Katharein near Troppau, and of 1084 at Castle Acre Priory, are all disconcerting to a beginner in the study of the question, and to have these dates distinctly rejected by a recognized authority on the subject is very helpful. On the positive side, to have at hand a set of tables giving the earliest authentic example of the numerals in the west, the well-known Codex Vigilanus of 976, and a score of other examples of the Boethian apices; to have the most characteristic specimens of the Arabic forms from the twelfth to the sixteenth century, not only as shown in the manuscripts, but also in the inscriptions and on the paintings of these periods; to be able at a glance to follow the lambda seven in its efforts to assume the upright form; to see the looped four giving place to the familiar form which finally replaced it; to see the beginning of the struggle of the zero for a place and for a recognized shape,—to have all this in scientific tabular form so that the changes can be traced at a glance, is to visit what may be described quite seriously as the cinematograph of our numeral system.

The tables, sixty-four in number and with hundreds of carefully drawn examples, are not without their surprises to those who have not looked into the subject. For example, the earliest known manuscript with the apices gives the upright

form of the seven instead of the lambda form which is commonly supposed to be the earlier one, and this is true, it may be said in passing, not only of the manuscript from which Mr. Hill has taken his illustration, but also the other Escorial manuscript of the same work. Likewise the upright four, which we ordinarily think of as due to the Florentines of the fifteenth century, who indeed had much to do with establishing it, is shown to have been used in the thirteenth and fourteenth centuries by English scribes and early in the fourteenth century by the Italians, probably Florentine monks, and quite commonly in the fifteenth century by writers of English manuscripts.

What strikes the reader as most gratifying is that Mr. Hill has brought to the problem a perfectly judicial mind; he has no thesis to defend; he is advocate for no party to any controversy; he is the scholar seeking absolute truth. To his researches, to his patience, to his care in weighing evidence, all who have an interest in the history of mathematics are quite as much his debtors as those whose fields of interest are in the lines of numismatics and paleography.

DAVID EUGENE SMITH.

Introduction to Infinitesimal Calculus. By G. W. CAUNT. Oxford, The Clarendon Press, 1914. xx+568 pp.

THIS book is an attempt to present the calculus in a way that will appeal to students of engineering. The author expresses a hope that he has made the book rigorous enough to satisfy the instructor in a first course in calculus for a student in pure mathematics. This seems to be rather an exception, most texts being written for the pure mathematician, or at least chiefly from his viewpoint. The subject matter is that usually found in the texts on calculus with the addition of a chapter on differential equations, and the author presents the subject from the viewpoint of the engineer. The book is written for a first course in calculus and is arranged for a minimum amount of analytic geometry to precede it. The author usually introduces a subject by means of a number of illustrative numerical examples worked out in detail, thus leading the student into a subject by means of his interest in the purpose it serves. This use of numerical examples, completely solved out, prepares the student of engineering to make use of his mathematics in his engineering courses. Too often a

student gets through his calculus without being able to apply it to numerical problems, especially when they occur in a course not designated as mathematics. This should not be so, and is undoubtedly the reason why mathematics in our technical schools is in such disfavor with the students. This situation is most probably due partly to the mathematics teachers, partly to the engineering teachers and partly to the textbooks. Wherever the fault lies, any textbook written for engineering students should bridge this gap or wholly close it. If there is any essential difference between the calculus for the engineer and the calculus for the pure mathematician, then our textbooks for engineers should be written as such and should not be attempts to compile books that can be sold to both classes of students. The teacher of mathematics in an engineering school who is seeking to present the calculus to his students in a way that will make it appeal to them as being a subject they need instead of one they must take will find this book a help in that direction.

The first two chapters, especially the second on "Limits and continuous functions," are an attempt to get the student familiar with subjects that often remain hazy until the end of his course in calculus. They present the matter in a very clear way by means of many examples with full explanations. The remaining chapters are treated in much the same way. The book is so arranged that a shorter course can be had by omitting certain chapters without destroying the continuity of presentation. The book contains more material than most of our engineering schools could cover in the time now allotted to mathematics. The author seems to have had liberty from his publishers to give as much space as he desired to illustrative problems and lists of exercises. This is a very good feature of the book. On the whole the book should prove very teachable.

T. E. MASON.

Die Rechenmaschinen und das Maschinenrechnen. Von Dipl. Ing. LENZ. Band 490, Sammlung aus Natur und Geisteswelt. Leipzig, Teubner, 1915. vi+114 pp.

OVER 500 separate numbers of this collection of booklets on science, the arts, and technology have been published and the set is not yet closed. Each volume is complete in itself and retails at M. 1.25.

No. 490—the one under review—aims to give its readers

clear and definite notions concerning the mathematical and mechanical principles underlying the construction of the many types of machines designed to perform automatically the operations of addition, subtraction, multiplication, and division.

No great mathematical or technical knowledge is required to read the book with ease; though it possesses much more of scientific interest and spirit than one might expect to find in a so-called " popular " treatise.

Representative machines, mostly of German or American make, varying in complexity from the abacus to the Burroughs —all designed to add or subtract—are described in detail, and the mechanical principles according to which they operate are discussed. Similarly there are separate chapters on machines designed to perform multiplication and division, the highest type of which is represented by the " millionaire " computing machine so often found in our mathematical laboratories. The text is illustrated by 45 excellent figures.

The author summarizes the present state of development to which mechanical computation has been carried; points out many imperfections which still exist, and suggests the requirements which the ideal machine should fulfill. He closes with a rather brief discussion of the principles underlying the construction and use of the slide rule. It seems to the reviewer that this chapter is rather inadequate and hardly up-to-date.

ERNEST W. PONZER.

Die mathematische Ausbildung der Deutschen Landmesser. Von PH. FURTWÄNGLER und G. RUHM. Band IV, Heft 8, I. M. U. K. Leipzig, Teubner, 1914. vi+50 pp.

IN this pamphlet is given a summary of the training, both practical and theoretical, together with the courses of study prescribed for the German engineer who wishes to specialize in land surveying. Though the various German states differ in their minimum requirements for the holder of this office, who must pass a rigid examination, yet nowhere are there evidences of the existence of an elective sinecure such as is represented by that of our own county surveyor, an office too often filled by some derelict engineer with a political pull. The work is systematized and is more of the grade of that of our Coast and Geodetic Survey.

Special courses are offered at the technical high schools in

the different states for the training of this class of engineers. The authors go into these in detail. A composite cross-sectional view of present practise would show about the following.

At least one or two years of practical training under the guidance of a regularly appointed surveyor. This practical experience may precede or follow the course of instruction in the technical schools. An average of about four semesters in a technical school offering the special courses required. An opportunity to advance in the profession. Advancement based solely on merit.

The courses offered necessarily include trigonometry, algebraic analysis, analytic and descriptive geometry, the calculus, mechanics, sometimes the method of least squares, astronomy, geodesy, drafting, map-making, seminar.

A general survey is made of the methods of handling these various subjects. And one is not surprised to find that throughout there are emphasized all those which aim to develop the order, accuracy, simplicity, and efficiency so desirable in any engineer.

Among suggestions intended to secure a greater efficiency the authors include such as the founding of higher schools, a three-year course of study, and a longer apprenticeship.

ERNEST W. PONZER.

Algebraic Invariants. By LEONARD EUGENE DICKSON. (No. 14, Mathematical Monographs, edited by Mansfield Merriman and Robert S. Woodward.) New York, John Wiley and Sons, 1914. x+100 pp.

IN this brief introduction to the classical theory of invariants Professor Dickson puts the reader under a further debt of gratitude for the excellent and entertaining way in which he is led to a first acquaintance with the important subject of invariants. It is difficult to conceive how one could be more comfortably drawn into a knowledge of invariants and covariants than by the gradual and lucid processes of the early part of this book. In the first ten pages there is a progressive approach to the full notion of invariant, carried forward from things well known to the beginner by means of processes and ideas which are of intrinsic interest and value in themselves. After a similar gradual development of the notion of covariant the formal definitions of invariants and covariants are given on pages 14 and 15.

The book falls into three parts of nearly equal length. Part I (pages 1–29) treats of linear transformations both from the standpoint of a change of points of reference and from the standpoint of projective geometry. Its purpose is to lead the reader by easy stages into a proper conception of the nature of the subject. How this purpose is achieved may be seen in part by the way in which the notions of invariant and covariant are introduced and illustrated by means of the simplest examples. Certain covariants such as Jacobians and Hessians are discussed and their algebraic and geometrical interpretations are given. In particular, the Hessian is employed in the solution of the cubic equation and in the discussion of the points of inflection of the plane cubic curve.

In connection with the interesting illustrative examples and applications of Part I the author finds opportunity to derive or illustrate several general elementary theorems so that the reader is making substantial progress in the theory as he learns its meaning through examples.

Part II (pages 30–62) is devoted to a systematic development, in non-symbolic notation, of the algebraic properties of invariants and covariants, chiefly of binary forms. In the preceding part the reader has been prepared so as to be able to follow this treatment with comfort. There is here a compact but lucid treatment of the following topics: homogeneity, weight, annihilators, alternants, seminvariant leaders of covariants, number of linearly independent seminvariants, Hermite's law of reciprocity, fundamental systems of covariants, canonical form of the binary quartic, properties of invariants and covariants as functions of the roots, and differential operators producing covariants. Moreover, irrational invariants are illustrated (in § 35) by a carefully selected set of exercises.

In part III (pages 63–97) is given an introduction to the symbolic notation of Aronhold and Clebsch. This notation is first explained at length for a simple case and the reader is led gradually into its use through a carefully constructed proof of the fundamental theorem on the types of symbolic factors of a term of a covariant of binary forms. That a fundamental system of covariants is finite is proved (pages 70–76) by a method due to Hilbert. The theory of transvectants is developed so far as needed in the treatment of apolarity and its application to rational curves and in the inductive deter-

mination of fundamental systems of covariants. Finally, there is a discussion of the concomitants of ternary forms in symbolic notation.

R. D. CARMICHAEL.

Leçons sur la Théorie des Fonctions. Par EMILE BOREL. Deuxième édition. Paris, Gauthier-Villars, 1914. xi+259 pp.

FOR purposes of review the second edition of this valuable book by Borel may be divided into three parts: the body of the text exclusive of the extensive notes at the end (Chapters I to VI, pages 1–101); notes contained in the first edition (pages 102–134); notes added in the second edition (pages 135–256). The first two parts are reprinted without modification except for a single change in a matter of terminology in the theory of point sets. Since these parts have been before the mathematical public for many years (having been originally published in 1898), they call for no further review now. The third part consists of three notes numbered IV, V, VI; it makes up about half of the present volume.

Note IV (pages 135–181) is devoted to polemics concerning the transfinite and the demonstration of Zermelo. It contains a reprint of seven articles, principally by Borel, published from time to time during the years 1899 to 1914. These discussions are perhaps of more interest to philosophers than to mathematicians. A perusal of this note in comparison with earlier statements by Borel shows that his thought on some of the matters in consideration has undergone a marked evolution.

Note V (pages 182–216) is devoted to denumerable probabilities and their arithmetic applications.

Finally, Note VI (pages 217–256) is given to a development of the theory of measure and of integration from the point of view adopted by Borel in his definition of measurable sets. It contains the most important matter added in this new edition of the work. The subject is approached in a very elementary and simple way and the treatment is carried through to results of considerable generality both in the theory of measure and in the theory of integration. The treatment will serve conveniently as an introduction to the fundamental researches of Lebesgue in the theory of integration.

R. D. CARMICHAEL.

Radioactive Substances and their Radiations. By E. RUTHERFORD, D.Sc., Ph.D., LL.D., F.R.S. Cambridge University Press, 1913. viii+699 pp.

THIS book contains a very interesting treatment of a fascinating subject by a writer who speaks with the authority of a leading investigator. Dealing effectively as it does with one of the most remarkable developments of modern physics, a science which has sprung anew into the most rapid growth in the present generation, it is assured of a wide range of readers and of a valuable place in the development of science. For the most part it deals with experimental researches and results and makes but a small use of mathematics in their interpretation. For this reason only a brief notice of it should be given here, although it appears to be a book of more than usual value.

The investigations of radioactivity are too new for the subject yet to have taken on a mathematical form. Mathematics is the dress in which a physical science is best expounded and developed after it has come to maturity and the fundamental laws involved in it have been determined with precision; but it ill becomes a subject while yet in the infancy of early experimental stages.

But in spite of this newness of the subject there is one mathematical investigation connected with it and of considerable interest owing to the differential equation to which it gives rise. This seems not to be mentioned by Rutherford. It was initiated by J. J. Thomson (see Encyclopaedia Britannica, 11th edition, volume 6, page 371) and further developed by Mie (*Annalen der Physik* (4) 13 (1904): 857–889) and others. In investigating the distribution of electric force when a current is passing through an ionized gas Thomson obtained a differential equation which Mie has transformed to

$$\frac{1-k^2}{2\lambda} z \frac{d^2z}{d\xi^2} - \frac{1-k^2}{4\lambda^2}\left(\frac{dz}{d\xi}\right)^2 + \frac{k}{\lambda}\frac{dz}{d\xi} - z + 1 = 0,$$

where k and λ are constants. In developing the theory of this equation considerable difficulty has arisen. A reading of the papers which have been devoted to it leaves one with the conviction that they do not satisfactorily dispose either of the mathematical question involved in the solution of the equation or of the electrical question of which this equation is in part the mathematical expression.

R. D. CARMICHAEL.

The Mathematical Analysis of Electrical and Optical Wave-Motion on the Basis of Maxwell's Equations. By H. BATEMAN, M.A., Ph.D. Cambridge University Press, 1915. vi +159 pp.

THIS book is intended as an introduction to certain recent developments of Maxwell's electromagnetic theory which are directly connected with the solution of the partial differential equation of wave motion. The higher parts of the theory which are based on the dynamical equations of motion are not considered.

In Chapter I (pages 1–24) the author starts from the fundamental equations for free ether in Maxwell's electromagnetic theory and shows in the first place that solutions of these equations may be obtained by means of solutions of the fundamental wave equation

$$(1) \qquad \Omega(u) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0,$$

where c is the velocity of light (and is taken to be a constant). A function u of x, y, z, t, satisfying equation (1), is called a wave function. In the second place the author exhibits a class of solutions of the fundamental equations of Maxwell by means of functions α and β (of x, y, z, t) of such a nature that if $F(\alpha, \beta)$ is an arbitrary function of α and β, F satisfies the partial differential equation

$$(2) \qquad \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2 = \frac{1}{c^2}\left(\frac{\partial F}{\partial t}\right)^2.$$

Continuing in Chapter I it is shown that the fundamental Maxwell equations for a material medium may be solved by means of functions u (of x, y, z) satisfying the equation

$$(3) \qquad \Delta u + k^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0,$$

the quantity k being a constant with respect to x, y, z, t. This equation is obviously satisfied by wave functions of the form

$$u = e^{\pm ikct} f(x, y, z).$$

In connection with a wave boundary whose equation may be expressed in the form $F(x, y, z, t) = 0$ equation (2) comes again into play and here expresses the fact that the moving wave

boundary moves normally to itself with the velocity of light.

The work under review is devoted principally to deriving the properties of functions satisfying equations (1) and (3) and to the applications of the resulting theory to problems of electrical and optical wave motion. It is desirable that all types of functions satisfying equations (1) and (3) should be studied and not merely those which admit readily of application to physical problems; for in this way a clearer light is thrown on the physical problems themselves. But there is a much more important reason why the scope of the inquiry should not be restricted. The theory of wave functions forms a natural extension of the theory of functions of a complex variable and may consequently lead to results of great value for the general theory of functions. Moreover, the theory of functions of two complex variables is closely connected with the theory of wave functions. Our author recognizes this breadth of range of interest in the theory of equations (1) and (3) and takes account of it in his exposition.

In Chapter II (pages 25–34) is to be found a general survey of the different methods of solving the wave equation, including the Bernoulli method of reduction to ordinary differential equations, the generalization of wave functions through the property of linearity of the wave equation and the method of transformations (developed by Bateman) by means of which other solutions may be obtained from a single given solution.

By means of a transformation to polar coordinates and a use of the Bernoulli method of reduction to ordinary differential equations various solutions of equations (1) and (3) are obtained in Chapter III (pages 35–68), and these are employed in the investigation of several important problems connected with spherical obstacles, including the following: scattering of electromagnetic waves by spherical obstacles; damped vibrations for the space outside of a sphere; polarization and intensity of the scattered light; absorption of light by a spherical obstacle; pressure of radiation on a spherical obstacle.

In Chapter IV (pages 69–81) a similar treatment is made of equations (1) and (3) by means of a transformation to cylindrical coordinates and the results are applied in a treatment of the propagation of waves on a semi-infinite solid bounded by a plane surface and also along a straight wire of circular cross section.

A partial solution of the problem of diffraction is obtained in Chapter V (pages 82–94) by means of multiple-valued solutions of the wave equation.

Chapter VI (pages 95–109) contains a brief account of each of several transformations of coordinates which lead to solutions of the wave equation suitable for the treatment of problems connected with surfaces of revolution. These give rise to several important ordinary differential equations of which it is desirable to have a more complete theory than we possess at present.

In Chapter VII (pages 110–114) methods are exhibited for finding homogeneous solutions of the wave equation.

Chapter VIII (pages 115–140) is given to a treatment of electromagnetic fields with moving singularities. It consists principally of the author's own contributions to the theory in consideration. From the investigations of this chapter it is seen that the mathematical analysis connected with equations (1) and (3) is suitable for the discussion of three distinct theories of the universe which may be briefly described as follows: (*a*) the ether is a continuous medium and matter consists of aggregates of discrete particles; (*b*) the ether is a discontinuous medium consisting of a collection of tubes or filaments; and matter is an aggregate of discrete particles attached to the tubes; (*c*) the ether is a continuous medium and matter is an aggregate of discrete particles to which tubes are attached. In Chapters I to VII the analysis is adapted almost entirely to the first of these theories, the high development of which we owe to the pioneer work of Maxwell, Fitzgerald, Hertz, Rayleigh, Heaviside, J. J. Thomson, Lorentz and Larmor. The other theories have heretofore not received much attention; and hence the contributions of Bateman (in Chapter VIII) form a welcome addition to our previous knowledge. It is to be hoped that they will lead to further developments so that a comparison can be made between the different theories. It is likely that each of the theories will be enriched by the development of the other two.

Finally, some miscellaneous theories are briefly treated in the concluding Chapter IX (pages 141–154).

This book will be found valuable for its new contributions to the theory of equations (1) and (3), for its account of the present state of knowledge and its numerous references to the literature of these equations and for its indication of several

points at which further development of the theory is desirable. Its greatest value probably lies in the way in which it makes clear that it is desirable to have a further and deeper investigation of the properties of wave functions and the wave equation.

R. D. CARMICHAEL.

NOTES.

THE Swiss mathematical society held its annual meeting at Geneva, September 12–15, in affiliation with the centenary celebration of the Helvetian society of natural scientists. The following papers were read: By Professor L. G. DU PASQUIER, "On systems of complex numbers"; by Dr. G. PÓLYA, "Is the non-continuation of a power series the general case?"; by Professor M. PLANCHEREL, "On the convergence of a remarkable class of definite integrals containing an arbitrary function"; by Dr. W. H. YOUNG and Dr. G. C. YOUNG, "Integration with regard to a function of limited variation"; by Dr. G. C. YOUNG, "On curves without tangents"; by Dr. K. MIRIMANOFF and Dr. G. C. YOUNG, "On the theorem of 'tuiles'"; by Professor L. J. CRELIER, "On a particular theorem of the geometry of motion"; by Professor R. DE SAUSSURE, "Geometry of leaflets"; by Professor CAILLER, "On the analytic theory of directed bodies"; by Dr. H. BERLINER, "A new projective analytic geometry"; by Professor L. KOLLROSS, "Concerning a duality"; by Dr. F. GONSETH, "Generalization of a theorem of Poncelet"; by E. GUILLAUME, "On the impossibility of reducing the law of divergence in several variables to a composite probability."

Professor M. GROSSMANN, of the technical school at Zurich, was elected president for the following year.

AT a meeting of the Edinburgh mathematical society on November 12 the following papers were read: "The solution of difference equations by continued fractions," by J. A. STRANG; "A suggested measurement of relationship" and "The equation $x^3 + y^3 + z^3 + u^3 = 0$ where x, y, z, u are rational," by J. E. A. STEGGALL; "Notes on a triangle," by G. E. CRAWFORD; "Easy geometrical proof of a theorem of Chasles," by E. PRESS.

The Association of mathematics teachers of New Jersey held its third regular meeting at Stevens institute of technology on November 20. The programme included the following papers: "Mathematics and insurance," by P. C. H. Papps; "The proper functioning of a high school course in geometry," by R. T. LeValley; "Review of Bourlet's Plane Geometry," by B. B. Strang; "A high school course in strength of materials," by G. D. Orner. Professor H. B. Fine is president of the association.

The December number (volume 17, number 2) of the *Annals of Mathematics* contains the following papers: "The Cauchy definition of a definite integral," by D. C. Gillespie; "Surfaces with isothermal representation of their lines of curvature as envelopes of rolling," by L. P. Eisenhart; "A theorem concerning real functions," by K. P. Williams; "Note on an operation of the third grade," by A. A. Bennett; "Determination of all triply orthogonal systems containing a family of minimal surfaces," by T. H. Gronwall.

The May number of the *Proceedings of the National Academy of Sciences* contains the following mathematical papers: "A new canonical form of the elliptic integral," by Bessie I. Miller; "Transformations of conjugate systems with equal invariants," by L. P. Eisenhart. The June number of the *Proceedings* contains: "Solution of an infinite system of differential equations of the analytic type," by F. R. Moulton; "On the factorization of various types of expressions," by Henry Blumberg. The July number contains: "On the representation of arbitrary functions by definite integrals," by W. B. Ford; "Some theorems connected with irrational numbers," by W. D. MacMillan. The August number contains "Seven points on a twisted cubic curve," by H. S. White, and the October number contains "On isothermally conjugate nets of space curves," by G. M. Green.

The Macmillan Company announces the publication of an Historical Introduction to Mathematical Literature, by G. A. Miller.

The firm of B. G. Teubner in Leipzig announces that the following works are in press: Euler's works, series 1, volume

2, part 1: Commentationes arithmeticae, by F. RUDIO; volume 18, part 2: Commentationes analyticae ad theoriam integralium pertinentes, by A. GUTZMER—Lie's collected memoirs, volume 3, by F. ENGEL—Vorlesungen über Zahlen- und Funktionenlehre, by A. PRINGSHEIM—The second part of the second volume of Darstellende Geometrie, by E. MÜLLER—Theorie der elliptischen Funktionen, by R. FRICKE —Vorlesungen über reelle Funktionen, by C. CARATHÉODORY —Differential- und Integralrechnung, by L. BIEBERBACH— Les théories statistiques en thermodynamique, by H. A. LORENTZ—Partial differential equations of mathematical physics, by A. G. WEBSTER—and a number of reports on the teaching of mathematics in Germany, edited by F. KLEIN.

THE following courses in mathematics are given during the present winter semester:

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Analytic geometry, four hours; Synthetic geometry, four hours; Elementary derivation of the most important properties of conics, two hours; Seminar, two hours.—By Professor G. FROBENIUS: Theory of numbers, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: Theory of curves and surfaces, four hours; General problems of the theory of functions, two hours; Seminar, two hours.—By Dr. K. KNOPP: Ordinary differential equations, II, four hours; Theory of functions, II, with exercises, five hours.—By Professor R. ROTHE: Integral calculus, four hours.

UNIVERSITY OF BONN.—By Professor E. STUDY: Analytic geometry of two and of three dimensions, four hours; Seminar, two hours.—By Professor F. LONDON: Differential and integral calculus, with exercises, five hours; Synthetic geometry, with exercises, three hours; Seminar, two hours.—By Professor I. SCHUR: Selected chapters of analysis, two hours; Introduction to the theory of functions, four hours; Seminar, two hours.—By Dr. J. O. MÜLLER: Calculus of variations, three hours; Seminar, two hours.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Development of mathematics in the nineteenth century, two hours.—By Professor D. HILBERT: Differential equations, four hours; Seminar, two hours.—By Professor C. RUNGE: Descriptive geometry, with exercises, five hours; Differential

and integral calculus, with exercises, six hours; Seminar, two hours.—By Professor E. LANDAU: Algebra, four hours; Seminar, two hours.—By Professor C. CARATHÉODORY: Curves and surfaces, four hours; Selected chapters from the theory of functions, with exercises, four hours.—By Professor F. BERNSTEIN: Mathematical statistics, two hours; Mathematics of insurance, two hours; Seminar, two hours.

UNIVERSITY OF LEIPZIG.—By Professor O. HÖLDER: Differential and integral calculus, five hours; Elliptic modular functions, two hours; Seminar, two hours.—By Professor K. ROHN: Analytic geometry of space, four hours; Differential geometry, four hours; Seminar, two hours.—By Professor G. HERGLOTZ: Mechanics, five hours; Selected chapters from the theory of numbers, three hours; Seminar, two hours.—By Dr. K. BLASCHKE: Partial differential equations, four hours; Conformal representation, two hours; Seminar, two hours.

UNIVERSITY OF STRASSBURG.—By Professor F. SCHUR: Analytic geometry of two and three dimensions, four hours; Selected chapters of differential geometry, two hours; Seminar, two hours.—By Professor G. FABER: Differential and integral calculus, four hours; Elliptic functions, two hours; Seminar, two hours.—By Professor M. SIMON: History of mathematics in ancient times, two hours.—By Professor J. WELLSTEIN: Graphical statics, three hours.—By Professor P. EPSTEIN: Analytic treatment of projective geometry, two hours.—By Dr. A. SPEISER: Perspective, two hours.

DR. J. RADON and DR. F. RULF have been appointed docents in mathematics at the technical school of Vienna.

DR. W. SIMANDL has been appointed docent in projective geometry at the Bohemian technical school of Brünn.

DR. K. BOPP, of the University of Heidelberg, has been promoted to an associate professorship of mathematics.

PROFESSOR G. FABER, of the University of Strassburg, has been appointed to a full professorship in the technical school at Munich.

PROFESSOR P. PAINLEVÉ, of the University of Paris, has been appointed minister of education in the new French cabinet.

Professor E. Pascal, of the University of Naples, has been awarded the medal of the Italian society of sciences (the forty).

Professor J. Mollerup has been appointed associate professor of mathematics in the University of Copenhagen, succeeding Professor H. Bohr, who has been called to the Copenhagen technical school as successor of the late Professor P. C. V. Hansen.

Dr. G. Armellini, of the University of Rome, has been appointed associate professor of mechanics at the technical school of Turin.

Professor M. Cipolla, of the University of Catania, has been promoted to a full professorship of algebraic analysis.

Dr. L. L. Rios has been appointed docent in mechanics at the University of Padua.

The two Benjamin Peirce instructorships in mathematics at Harvard University (see Bulletin, volume 21, page 315) are again open to general competition. Applications for the year 1916–17, accompanied by the necessary papers, should reach Professor Bôcher not later than February 1, 1916, who will be glad to furnish further particulars.

Professor N. C. Grimes, formerly of the University of Arizona, has been appointed to a professorship of mathematics in the University of Oregon.

Miss Gertrude I. McCain has been appointed professor of mathematics in Oxford College, not in the Western College for Women as erroneously announced in the December Bulletin. The professorship of mathematics in the Western College for Women is held by Miss Harriet E. Glazier.

At the State University of Kentucky associate professor J. M. Davis has been promoted to a full professorship of mathematics.

Miss A. D. Lewis, of Mt. Holyoke College, has been appointed professor of mathematics and head of the department at the Kentucky College for Women.

At the Iowa State College the following changes have been made in the department of mathematics: Miss A. Fleming has been promoted to an assistant professorship; Mr. J. R. Sage, Miss F. Farnum, and Miss M. Miller have been appointed instructors; Miss N. Madson has been appointed assistant.

Professor H. L. Rietz, of the University of Illinois, has been appointed by Governor Dunne a member of the committee to investigate the operation of national and foreign pension systems.

Professor W. H. H. Hudson, of King's College, London, died September 21, at the age of 76 years.

Professor E. Janisch, of the technical school at Prague, died August 11, 1915, at the age of 46 years.

Professor J. Knoblauch, of the University of Berlin, died July 22, 1915, at the age of 59 years.

Professor F. Leconte, of the University of Ghent, died October 11, 1915, at the age of 50 years.

Professor H. Ganter, of the cantonal school at Aargau, died July 29, 1915, at the age of sixty-seven years.

Dr. L. Orlando, of the University of Rome, was killed in battle in August.

Dr. R. Torelli, of the University of Pisa, was killed in battle August 10, 1915, at the age of thirty-one years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Berzolari (L.). See Encyclopédie.

Besserve (A.). Le cercle et les surfaces cerclées en géométrie conforme. (Thèse.) Paris, Gauthier-Villars, 1915. 4to. 8+162 pp. Fr. 9.00

Bôcher (M.). Plane analytic geometry. New York, Holt, 1915. 12mo. 13+235 pp. $1.60

Brown (E. H.). See Burn (J.).

Burn (J.) and Brown (E. H.). Elements of finite differences; also solutions to questions set for part 1 of the examinations of the Institute of actuaries. London, Layton, 1915. 10s. 6d.

CARRUS (S.). See ENCYCLOPÉDIE.

CARSE (G. A.) and SHEARER (G.). A course in Fourier's analysis and periodogram analysis for the mathematical laboratory. (Edinburgh Mathematical Tracts No. 4.) London, Bell, 1915. 8vo. 8+66 pp. 3s. 6d.

CARTAN (E.). See ENCYCLOPÉDIE.

CHARLES (R. L.). See FRANKLIN (W. S.).

DINGELDEY (F.). See ENCYCLOPÉDIE.

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome III, volume 1, fascicule 2: Les notions de ligne et de surface par H. von Mangoldt et L. Zoretti; Exposé parallèle du développement de la géométrie synthétique et de la géométrie analytique pendant le 19ième siècle par G. Fano et S. Carrus; Géométrie énumérative par H. G. Zeuthen et M. Pieri; La théorie des groupes continus et la géométrie par G. Fano et E. Cartan. Paris, Gauthier-Villars et Leipzig, Teubner, 1915. Gr. 8vo. Pp. 161–352. M. 7.20

——. Tome III, volume 3, fascicule 2: Systèmes de coniques par F. Dingeldey et E. Fabry; Théorie générale des courbes planes algébriques par L. Berzolari. Paris, Gauthier-Villars et Leipzig, Teubner, 1915. Gr. 8vo. Pp. 161–304. M. 5.40

FABRY (E.). See ENCYCLOPÉDIE.

FANO (G.). See ENCYCLOPÉDIE.

FORD (L. R.). An introduction to the theory of automorphic functions. (Edinburgh Mathematical Tracts No. 6.) 8vo. 8+96 pp. 3s. 6d.

FRAENKEL (A.). Ueber die Teiler der Null und die Zerlegung von Ringen. (Dissertation.) Marburg, 1914.

FRANKLIN (F. W.). Thoughts on ultimate problems. London, David Nutt, 1915. 150 pp. 1s. 6d.

FRANKLIN (W. S.), MACNUTT (B.) and CHARLES (R. L.). Calculus supplement. To take the place of pages 1–41 of the 1913 edition. South Bethlehem, Pa., New Era Printing Co., 1915. 8vo. 51 pp. Paper.

GIBB (D.). A course in interpolation and numerical integration for the mathematical laboratory. (Edinburgh Mathematical Tracts No. 2.) London, Bell, 1915. 8vo. 8+90 pp. 3s. 6d.

GLENN (O. E.). Treatise on the theory of invariants. Boston, Ginn, 1915. 8vo. 10+245 pp. Cloth. $2.75

KAMPÉ DE FÉRIET (J.). Sur les fonctions hypersphériques. (Thèse.) Paris, Gauthier-Villars, 1915. 4to. 112 pp. Fr. 6.00

KEYSER (C. J.). Science and religion: the rational and the supernatural. Address delivered before the Phi Beta Kappa Association in the City of New York. New Haven, Yale University Press, 1914. 12mo. 75 pp. $0.75

——. The new infinite and the old theology. New Haven, Yale University Press, 1915. 12mo. 117 pp. $0.75

KÖNNEMANN (W.). Rationale Lösungen von Aufgaben aus dem Gebiete der gesamten Elementarmathematik in funktionaler Abhängigkeit. Berlin, Winckelmann, 1915. 112 pp. M. 3.00

MACNUTT (B.). See FRANKLIN (W. S.).

MANGOLDT (H. VON). See ENCYCLOPÉDIE.

PÉRÈS (J.). Sur les fonctions permutables de première espèce de M. Vito Volterra. (Thèse.) Paris, Gauthier-Villars, 1915. 4to. 100 pp. Fr. 5.00

PIERI (M.). See ENCYCLOPÉDIE.

PRANGE (G.). Die Hamilton-Jacobische Theorie für Doppelintegrale (mit einer Uebersicht der Theorie für einfache Integrale). (Dissertation.) Göttingen, 1915.

SHEARER (G.). See CARSE (G. A.).

SCHMIDT (M. C. P.). Kulturhistorische Beiträge zur Kenntnis des römischen Altertums. 1tes Heft: Zur Entstehung und Terminologie der elementaren Mathematik. 2te Auflage. Leipzig, Dürr, 1914. 269 pp. M. 5.00

TOWNSEND (E. J.). Functions of a complex variable. (American mathematical series.) New York, Holt, 1915. 7+384 pp. $4.00

WILDBRETT (A.). Analytische Geometrie und Elemente der Differentialrechnung. Lehrbuch mit Aufgabensammlung für die Oberklasse von Gymnasien und Realgymnasien. Nürnberg, Korn, 1915. 124 pp. M. 2.00

ZEUTHEN (H. G.). See ENCYCLOPÉDIE.

ZORETTI (L.). See ENCYCLOPÉDIE.

II. ELEMENTARY MATHEMATICS.

BELL (H.). A course in the solution of spherical triangles for the mathematical laboratory. (Edinburgh Mathematical Tracts No. 5.) London, Bell, 1915. 8vo. 8+66 pp. 2s. 6d.

BEINHORN (H.). Lehrbuch der Mathematik. Berlin, Weidmann, 1915. Ausgabe A für Realanstalten, 4 Bände. 165+197+216+234 pp. M. 2.00+2.20+2.40+2.80. Ausgabe B für Gymnasialanstalten, 3 Bände. 274+175+211 pp. M. 3.00+2.20+2.40.

BORCHARDT (W. G.). Revision papers in algebra. London, Rivingtons, 1915. 2s.

BORCHARDT (W. G.) and PERROTT (A. D.). Key to geometry for schools. London, Bell, 1915. 294 pp. 8s. 6d.

CRANTZ (P.). Arithmetische Aufgaben für Oberlyzeen sowie die mittleren und oberen Klassen der Studienanstalten. 2te Auflage. Leipzig, Teubner, 1913. 88 pp. M. 1.40

DUPUIS (N. F.). Elementary synthetic geometry of the point line and circle, in the plane. 4th reprint. New York, Macmillan, 1914. 290 pp. $1.10

——. Elements of synthetic solid geometry. 3d reprint. New York, Macmillan, 1914. 12mo. 239 pp. $1.60

FISHER (G. E.) and SCHWATT (I.). Complete secondary algebra. 3d reprint. New York, Macmillan, 1914. 12mo. 504 pp. $1.35

GERBER (L.). Rechenunterricht und Krieg. Strassburg, Strassburger Druckerei und Verlagsanstalt, 1915. 80 pp. M. 1.00

GNAGA (A.). Sulla estrazione di radice quadrata e cubica col procedimento delle medie. Brescia, tip. F. Apollonio, 1914. 8vo. 15 pp.

HUN (J. G.) and MACINNES (C. R.). The elements of plane and spherical trigonometry. 3d reprint. New York, Macmillan, 1915. $0.90

INGOLD (L.). See KENYON (A. M.).

KENYON (A. M.) and INGOLD (L.). Plane and spherical trigonometry. 2d reprint. New York, Macmillan, 1914. 132 pp. $1.35

LESSER (O.). See SCHWAB (K.).

LISTER (S.). A first book of arithmetic. London, Macmillan, 1915. 28+239 pp. 1s. 6d.

MACINNES (C. R.). See HUN (J. G.).

MILNE (R. M.). Mathematical papers for admission into the Royal Military Academy and the Royal Military College, February–June, 1915. London, Macmillan, 1915. 8vo. 24 pp. 1s.

PERROTT (A. D.). See BORCHARDT (W. G.).

ROBBINS (E. R.). New plane geometry. New York, American Book Co., 1915. 6+264 pp.

ROTHROCK (D. A.). Elements of plane and spherical trigonometry. 7th reprint. New York, Macmillan, 1915. 147 pp. $1.10

SAMMLUNG mathematischer Formeln. Herausgegeben vom bayerischen Mathematiker-Verein. Ausgabe A für Gymnasien, Realgymnasien, Realschulen. Ausgabe B für Oberrealschulen. München, J. Lindauer, 1915.

SCHNEIDER (A.). See SCHWAB (K.).

SCHULTZE (A.). Advanced algebra. 12th reprint. New York, Macmillan, 1915. 12mo. 562 pp. $1.25

SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk: A. Schneider, Lehr- und Uebungsbuch der Geometrie. Für Lehrer- und Lehrerinnenbildungsanstalten. 2ter Teil (Ebene Trigonometrie, Stereometrie und sphärische Trigonometrie). Leipzig, Freytag, 1915. 262 pp. M. 3.40

SCHWATT (I.). See FISHER (G. E.).

SMITH (C.). Elementary algebra. Complete edition, revised and adapted to American schools by I. Stringham. 5th reprint. New York, Macmillan, 1915. 12+672 pp. $1.20

STRINGHAM (I.). See SMITH (C.).

STUCKEY (J. J.). Table of compound interest at $\frac{1}{8}$ per cent. and of antilogarithms to sixty figures to base 1.00125. London, G. Allen and Unwin, 1915. 4to. 116 pp. 1£ 1s.

TABLES for converting shillings and pence and farthings into 7 places of decimals of a pound and for the reconversion of decimals. London, Layton, 1915. 8 pp. 1s.

THAER (A.) und WIMMENAUER (T.). Arithmetische Aufgaben für höhere Schulen. Ausgabe B, für Oberrealschulen, Realgymnasien und verwandte Anstalten. 1ter Teil: Unterstufe; 2ter Teil: Oberstufe. Leipzig, Hirt, 1915. 146+152 pp. M. 1.75+1.75

URAGUCHI (Y.). Handy logarithmic tables. Tokyo, Y. Uraguchi, 1915. 12 sen

WENZEL (G.). Geometrie (Hilfsbücher zur Vorbereitung für die Bürgerschullehrerprüfung, V. Band). Wien, Tempsky, 1915. 404 pp. K. 5.00

WIMMENAUER (T.). See THAER (A.).

ZICHEROW (—.). Das abgekürzte Rechnen, für Eltern und Schüler dargestellt. Rawitsch, Birkenstock, 1915. 15 pp. M. 0.35

III. APPLIED MATHEMATICS.

ABETTI (A.). Osservazioni astronomiche fatte all'equatoriale di Arcetri nel 1914, ed appendice di M. Maggini. Firenze, tip. Galletti e Cocci, 1915. 4to. 105 pp.

ANDERSON (A. J.). The romance of Leonardo da Vinci. New York, Brentano, 1915. $3.75

ANNUARIO astronomico pel 1916, pubblicato dal r. osservatorio di Pino Torinese. Torino, tip. s. Giuseppe degli Artigianelli, 1915. 8vo. 165 pp. +1 tavola.

BARTON (E. H.). An introduction to the mechanics of fluids. London, Longmans, 1915. 14 +249 pp. 6s.

BIGOURDAIN (G.). Petit atlas céleste précédé d'une introduction sur les constellations, sur les moyens de les reconnaitre, . . . Paris, Gauthier-Villars, 1915. 60 pp. +5 cartes. Fr. 2.75

BLUMENTHAL (O.). See BORN (M.).

BORN (M.). Dynamik der Kristallgitter. (Fortschritte der mathematischen Wissenschaften in Monographien. Herausgegeben von O. Blumenthal. Heft 4.) Leipzig, Teubner, 1915. Gr. 8vo. 7 + 122 pp. M. 7.60

CASSINIS (G.). See REINA (V.).

CONNAISSANCE des temps ou des mouvements célestes pour le méridien de Paris à l'usage des astronomes et des navigateurs pour 1917 publiée par le Bureau des longitudes. Paris, Gauthier-Villars, 1915. 8vo. 30 +824 pp. Cartonné. Fr. 4.75

CONWAY (A. W.). Relativity. (Edinburgh Mathematical Tracts No. 3.) London, Bell, 1915. 8vo. 8 +43 pp. 2s.

COOK (T. A.). The curves of life, being an account of spiral formations and their application to growth in nature, to science and to art; with special reference to the manuscripts of Leonardo da Vinci. New York, Holt, 1915. Gr. 8vo. 30 +479 pp. $5.00

DARIÈS (G.). Cubature des terrasses et mouvement des terres. 2e édition. Paris, Gauthier-Villars, 1914. 8vo. 196 pp. Cartonné. Fr. 3.00

DREYER (I. L. E.). See TYCHONIS BRAHE.

GREENHILL (G.). Report on gyroscopic theory. (Advisory committee for aeronautics, Reports and memoranda No. 146.) London, Darling and Son, 1914. Folio. 277 pp. Boards. 10s.

GREGORY (R. A.) and HADLEY (H. E.). A manual of mechanics and heat. London, Macmillan, 1915. 8 +309 pp. 3s.

GROAT (B. F.). Chemi-hydrometry and its application to the precise testing of hydro-electric generators. (*Proceedings of the American Society of Civil Engineers*, vol. 41, No. 9, pp. 2103–2427.) New York, 1915.

HADLEY (H. E.). See GREGORY (R. A.).

HAUSSNER (R.). Darstellende Geometrie. 2ter Teil: Perspektive ebener Gebilde; Kegelschnitte. 2te verbesserte und vermehrte Auflage. Berlin, Göschen, 1914. M. 0.90

HIGBEE (F. G.). The essentials of descriptive geometry. New York, Wiley, 1915. 6+204 pp. $2.00

INCE (E. L.). A course of descriptive geometry and photogrammetry for the mathematical laboratory. (Edinburgh Mathematical Tracts No. 1.) London, Bell, 1915. 8vo. 8+79 pp. 2s. 6d.

JOHNSON (J. F.). Practical shop mechanics and mathematics. New York, Wiley, 1915. 9+130 pp. $1.25

KOPPE (M.). Die Bahnen der beweglichen Gestirne im Jahre 1915. Eine astronomische Tafel nebst Erklärung. Berlin, 1915. M. 0.40

LEONARDO DA VINCI. See COOK (T. A.)

MAGGINI (M.). See ABETTI (A.).

MELINAT (G.). Astronomische Schullektionen. Leipzig, Dürr, 1914. 62 pp. M. 1.60

MURANI (O.). Proprietà cardinali dei sistemi diottrici; strumenti d'ottica. (Biblioteca tecnica.) Milano, Hoepli, 1915. 8vo. 11+266 pp. L. 6.50

ORR (M. A.). Stars of the southern skies. London, Longmans, 1915. 12+92 pp. 2s.

PEASE (C. A.). First year course in general science. New York, Merrill, 1915. 315 pp. $1.20

POPPLEWELL (W. C.). The elements of surveying and geodesy. London, Longmans, 1915. 8vo. 12+244 pp. 7s. 6d.

REINA (V.) e CASSINIS (G.). Determinazioni di latutudine astronomica e di gravità relativa eseguita in Umbria e in Toscana nel 1913 (r. Commissione geodetica italiana). Roma, tip. Nazionale, di G. Bertero, 1915. 4to. 58 pp.

RYAN (W. T.). Continuous and alternating current machinery problems. New York, Wiley, 1915. 37 pp. $0.65

SEYRIG (T.). Statique graphique des systèmes triangulés. I: Exposés théoriques. 3e édition. Paris, Gauthier-Villars, 1915. 21 planches. Cartonné. Fr. 3.00

STORMER (C.). The solar eclipse of August 21, 1914. Kristiania, J. Dybwad, 1915. 6 pp. +4 plates.

TAYLOR (E. H.). Mathematics in the lower and middle commercial and industrial schools of various countries represented in the International commission on the teaching of mathematics. (United States Bureau of Education Bulletin, 1915, No. 35.) Washington, Government Printing Office, 1915. 96 pp. $0.15

THOMPSON (S. P.). Elementary lessons in electricity and magnetism. New edition completely revised and in many parts rewritten. London, Macmillan, 1915. 8vo. 4s. 6d.

TYCHONIS BRAHE Dani opera omnia edidit I. L. E. Dreyer. Tomus 2. Hauniae, Libraria Gyldendaliana, 1915. 461 pp.

VENTURA (T.). Tavole tacheometriche centesimali complete; calcolo delle coordinate piane rettangolari; riduzione all'orizzonte delle distanze lette sulla stadia verticale; calcolo delle differenze del livello. Torino, tip. Doyen, di L. Simondetti, 1915. 4to. 11+12 pp. L. 3.50

WEATHERHEAD (R.). The star pocket-book: or how to find your way at night by the stars. London, Longmans. 92 pp. 1s.

THE NINTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

The ninth regular meeting of the Southwestern Section of the Society was held at Washington University, St. Louis, Missouri, on Saturday, November 27, 1915. Thirty-eight persons attended the sessions, including the following twenty-one members of the Society:

Professor L. D. Ames, Mr. Charles Ammerman, Professor E. W. Davis, Dr. W. W. Denton, Professor E. P. R. Duval, Dr. E. A. Engler, Professor A. B. Frizell, Professor E. R. Hedrick, Professor G. O. James, Professor O. D. Kellogg, Mr. J. C. Rayworth, Professor S. W. Reaves, Professor W. H. Roever, Dr. H. M. Sheffer, Professor C. H. Sisam, Professor H. E. Slaught, Professor E. J. Townsend, Professor J. N. Van der Vries, Professor C. A. Waldo, Dr. Eula A. Weeks, Professor W. D. A. Westfall.

The morning session opened at 10 A.M. and the afternoon session at 2 P.M. Professor Roever presided. It was decided to hold the next meeting of the Section at the University of Kansas, Lawrence, Kansas, on Saturday, December 2, 1916. The following program committee was appointed: Professor J. N. Van der Vries (chairman), Professor S. W. Reaves, Professor O. D. Kellogg (secretary). Attending members were entertained at a smoker at the Washington Hotel on Friday evening, and at luncheon at the Tower Dormitory on Saturday noon.

The following papers were presented at this meeting:

(1) Professor C. H. SISAM: "On sextic surfaces which have a nodal curve of order eight."

(2) Professor G. H. HARDY: "Weierstrass's non-differentiable function."

(3) Professors E. R. HEDRICK and LOUIS INGOLD: "Note on the continuity of the function ξ in the law of the mean."

(4) Dr. S. LEFSCHETZ: "On the n-dimensional cycles of an algebraic n-dimensional variety."

(5) Professor A. B. FRIZELL: "The postulate of time."

(6) Professor K. P. WILLIAMS: "A theorem on real functions."

(7) Professor W. H. ROEVER: "Mathematical theory of

the optical phenomenon observed in viewing a light through a screen."

(8) Dr. C. H. Forsyth: "A method of interpolating single values among groups of values."

(9) Professor S. W. Reaves: "Metric properties of flecnodes on ruled surfaces."

(10) Professor H. C. Gossard: "Note on the Euler line."

(11) Mr. J. C. Rayworth: "On the generation of epicycloidal and hypocycloidal curves."

(12) Professor W. H. Roever: "Graphical construction for a function of a function and for a function parametrically given."

(13) Professor O. D. Kellogg: "On the roots of minors in the secular equation."

(14) Professor M. B. Porter: "On Savary's construction for the centers of curvature of a roulette."

In the absence of their authors, the papers of Professor Hardy, Dr. Lefschetz, Professor Williams, Dr. Forsyth, Professor Gossard, and Professor Porter were read by title. Abstracts of the papers follow.

1. In this paper, Professor Sisam points out some fundamental properties of sextic surfaces which have a nodal curve of order eight, of the sextic surfaces in space of five dimensions of which they are the projections, and of certain line congruences associated with them. The paper will appear in the *American Journal of Mathematics.*

2. It was proved by Weierstrass that the function

$$f(x) = \Sigma\, b^n \cos a^n \pi x,$$

where $0 < b < 1$ and a is an integer, has no differential coefficient for any value of x if $ab > 1 + \frac{3}{2}\pi$. The last condition is evidently artificial. In Professor Hardy's paper Weierstrass's function is discussed by a new method which leads to much more general and natural results. In particular it is shown that the function (and the corresponding function defined by a series of sines) has no finite differential coefficient for any value of x if $ab > 1$. The restriction that a is an integer is shown to be unnecessary. Similar results are established for other classes of functions. This paper will appear in the *Transactions.*

3. In this note, Professors Hedrick and Ingold show that the function $\xi(h)$ in the equation $[f(a + h) - f(a)]/h = f'(\xi)$ is continuous for h in the closed interval from 0 to $b - a$ provided there is, for each value of h in the interval, a single number ξ between a and b satisfying the above equation.

Incidentally, under the same hypothesis, certain other properties of the derivative and of the difference quotient are obtained.

4. In this paper, Dr. Lefschetz extends and generalizes a mode of generation of superficial cycles of an algebraic surface obtained by Emile Picard (Picard-Simart, Traité des Fonctions algébriques de deux Variables, volume 2, page 335), largely by the use of topological methods. The method followed in this note is different in that (a) only an infinitesimal deformation is used, and (b) it is based upon the value of Picard's invariant ρ_0. The generalization, which by Picard's method is arduous at best, is thus made comparatively simple. The paper is to appear in the *Rendiconti del Circolo Matematico di Palermo*.

5. Discussions on Mengenlehre have put in evidence two opposing tendencies; one is represented by the formal reasoning of Zermelo, the other finds remarkably clear expression in the philosophy of Bergson. To Bergson reality is the experience of change. Therefore the intuition of time, or duration as he prefers to call it, is essential to reality. Zermelo in his Auswahlprincip sets up an axiom which does not need duration and can not be tested by experience. By aid of a scheme described in the November BULLETIN, page 71, Professor Frizell develops a process for producing well-ordered sets by successive steps which taken singly may be described in terms of duration but constitute a sequence that overlaps duration. It follows that for mathematics time is not an *a priori* intuition; it is only a postulate which distinguishes intuitionism from formalism.

6. The theorem given by Professor Williams is a generalization of the theorem that a continuous function actually assumes all values between any two of its values. The paper has appeared in full in the December number of the *Annals of Mathematics*.

7. When a source of light is viewed through a screen certain curves of light become visible. Professor Roever shows that these curves, of which there are three, are the loci of the brilliant points (images of light) of the wires of the screen and that, in general, they are cubics. For an infinitely distant source of light they are conics and, in any case, they are nearly straight in the neighborhood of the light, i. e., of the point in which the plane of the screen is pierced by the ray from the light to the eye, through which point they all pass. One of the curves is the locus of the brilliant points of the practically straight cross wires of the screen. The other two are the loci of the brilliant points of the lengthwise wires, which bend in and out around the cross wires. When the ray from the light to the eye is perpendicular to the plane of the screen only two curves are seen (at least in the neighborhood of the point of crossing) and these cut at right angles. Geometrical constructions are given for all of the curves.

8. Dr. Forsyth's method of interpolation can be explained best by an illustration: Given the following age group of deaths, (5–9) 4129, (10–14) 2617, (15–19) 4317, the method gives the number of deaths for any single age, all computation being based upon the differences of the age groups. Dr. Forsyth incidentally gives the leading term and differences for interpolating several (such as five) values at a time. Thus, a group, such as given above, may be broken up completely into the several single values. The paper will be offered to the *Journal of the Royal Statistical Society* (England).

9. Professor Reaves in this paper makes use of the Wilczynski projective theory of ruled surfaces to study some metric properties of such surfaces. These relate chiefly to the flecnodes, flecnode curves, osculating quadric, and the locus of the center of the osculating quadric.

10. Euler proved that orthocenter, circumcenter, and centroid of a triangle are collinear, and the line through them has received the name "Euler line." He also proved that the Euler line of a given triangle together with two of its sides forms a triangle whose Euler line is parallel with the third side of the given triangle. By the use of vector coordinates or ordinary projective coordinates, Professor Gossard

proves the following theorem: the three Euler lines of the triangles formed by the Euler line and the sides, taken by twos, of a given triangle, form a triangle triply perspective with the given triangle and having the same Euler line. The orthocenters, circumcenters and centroids of these two triangles are symmetrically placed as to the center of perspective.

11. In Mr. Rayworth's paper, the number of ways of generating any epicycloidal or hypocycloidal curve, the locus of a point on the nth rolling circle, is shown to be $(n+1)!$ Since the arcs of the circumferences rolled over are arbitrary multiples of the preceding arcs, a dependence exists between them which finds expression thus: the sum of the products of the angles in the terms of the equations by their respective coefficients vanishes identically. Similar results are found when the fixed circle is replaced by a straight line. Some particular cases are considered.

12. A simple construction, suggested by the methods of descriptive geometry, for the graphical representation of a function of a function, and of a function given by two parametric equations is given in Professor Roever's second paper. The paper will appear in the *American Mathematical Monthly.*

13. In a problem connected with integral equations, it is of interest to know about the signs of the first minors of the symmetric matrix $[a_{ij} - \delta_{ij}\rho]$ for the roots of the secular equation $|a_{ij} - \delta_{ij}\rho| = 0$, where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. It is known that any set of principal minors of this matrix, each of which is a first minor of the preceding, forms a Sturm sequence. Professor Kellogg shows that if all the minors of $[a_{ij}]$ are $\geqq 0$, the functions $|a_{ij} - \delta_{ij}\rho|$, and the minors of the elements of its first row, $A_{11}(\rho)$, $A_{12}(\rho)$, $\cdots$, $A_{1n}(\rho)$, form, for positive values of ρ, a Sturm sequence. This gives information on the signs of all first minors, because of the proportionality subsisting among first minors of a vanishing determinant.

14. The purpose of Professor Porter's paper is to show how Savary's elegant construction can be derived by the most elementary considerations from projective geometry. The paper will be offered to the *American Mathematical Monthly.*

O. D. KELLOGG,
Secretary of the Section.

A NOTE ON THE PROBLEM OF LAGRANGE IN THE CALCULUS OF VARIATIONS.

BY PROFESSOR GILBERT AMES BLISS.

(Read before the American Mathematical Society, December 31, 1915.)

THERE are two theorems concerning the solutions of a system of linear differential equations, due to von Escherich and A. Mayer,* which play an important rôle in the proofs of the sufficient conditions for a problem of the calculus of variations in the form proposed by Lagrange. Bolza remarks† that the theory of the second variation has so far been essential to the establishment of these two theorems, as well as to the proof of the necessity of Jacobi's condition in the exceptional cases not covered by the geometrical theory of Kneser. In a paper which will appear in the near future, the writer has shown that for problems in parametric form in any number of dimensions an inclusive proof of the Jacobi condition may be made very simply by an application of Euler's equations and the usual corner point condition to the second variation. A similar result has been attained for the problem of Lagrange by D. M. Smith. It is desirable therefore to have deductions of the two theorems mentioned above which also shall be independent of the complicated transformations of the second variation. The proofs given below have this advantage.

§ 1. *The Differential Equations.*

The form‡

$$2\Omega(\eta, \eta', \mu) = \sum_{i,k} (P_{ik}\eta_i\eta_k + 2Q_{ik}\eta_i\eta_k' + R_{ik}\eta_i'\eta_k') + \sum_{\beta, i} \mu_\beta (S_{\beta i}\eta_i + T_{\beta i}\eta_i'),$$

where i, k, β have the ranges

$$i, k = 1, 2, \cdots, n; \quad \beta = 1, 2, \cdots, m; \quad m < n,$$

is quadratic and homogeneous in the variables η, η', μ, with coefficients and variables which are supposed to be functions of x of class C' on an interval $x_1 \leqq x \leqq x_2$. The prime denotes

* See Bolza, Vorlesungen über Variationsrechnung, p. 633.
† Loc. cit., p. 634.
‡ Bolza, loc. cit., p. 621.

differentiation with respect to x. In the sequel it will be convenient to represent multipartite numbers and matrices by single symbols, with the usual agreements as to the meanings of their products.* The expression for Ω would then be

$$2\Omega(\eta, \eta', \mu) = P\eta\cdot\eta + 2Q\eta\cdot\eta' + R\eta'\cdot\eta' + 2S\eta\cdot\mu + 2T\eta'\cdot\mu.$$

The matrices P and R are symmetric.

The system of $m + n$ linear differential equations to be considered can now be represented in the form

$$(1) \qquad \Phi(\eta) = \Omega_\mu = 0, \quad \Psi(\eta, \mu) = \Omega_\eta - \frac{d}{dx}\Omega_{\eta'} = 0,$$

where the symbol Ω_μ, for example, represents the multipartite number $(\partial\Omega/\partial\mu_1, \cdots, \partial\Omega/\partial\mu_m)$. A multipartite number is by definition equal to zero only when every element is zero. There are therefore m of the former equations and n of the latter.

The quadratic form Ω satisfies the well-known relations

$$(2) \qquad \begin{aligned} &\eta\cdot\Omega_\eta + \eta'\cdot\Omega_{\eta'} + \mu\cdot\Omega_\mu = 2\Omega, \\ &\eta\cdot\Omega_u + \eta'\cdot\Omega_{u'} + \mu\cdot\Omega_\sigma = u\cdot\Omega_\eta + u'\cdot\Omega_{\eta'} + \sigma\cdot\Omega_\mu, \end{aligned}$$

where (u, σ) is a second set of arguments of the type (η, μ), and Ω_u, for example, is the row of derivatives of the function $\Omega(u, u', \sigma)$ with respect to the elements of u. The product, indicated by a dot, of two multipartite numbers such as η and Ω_u, is the sum of the products of their respective elements. As a result of the second of the relations (2) one finds the formula

$$u\cdot\Psi(\eta, u) - \eta\cdot\Psi(u, \sigma) + \sigma\cdot\Phi(\eta) - \mu\cdot\Phi(u) = \frac{d}{dx}(\eta\cdot\Omega_{u'} - u\cdot\Omega_{\eta'}).$$

* The notations and properties used here are very simple. The symbols η, η', μ, for example, denote the multipartite numbers

$$\eta = (\eta_1, \eta_2, \cdots, \eta_n), \quad \eta' = (\eta_1', \eta_2', \cdots, \eta_n'), \quad \mu = (\mu_1, \mu_2, \cdots, \mu_m),$$

and Q is the matrix of elements Q_{ik} $(i, k = 1, 2, \cdots, n)$. Then $Q\eta$ is by definition a multipartite number

$$Q\eta = (\Sigma Q_{1k}\eta_k, \Sigma Q_{2k}\eta_k, \cdots, \Sigma Q_{nk}\eta_k),$$

and $Q\eta\cdot\eta'$ is the bilinear expression found by multiplying each element of $Q\eta$ by the corresponding element of η' and adding, according to the usual definition of scalar multiplication. If further details seem to be necessary for the developments of the present paper, see Bliss, "The solutions of differential equations of the first order as functions of their initial values," *Annals of Mathematics*, 2d series, vol. 6 (1905), p. 58.

Every pair of solutions (η, μ), (u, σ) of the equations (1) clearly makes the last expression in parenthesis have a constant value. When this constant is zero the two solutions are said to be *conjugate*.

A *conjugate system* of solutions is a system of n linearly independent solutions every two of which are conjugate. The functions u, σ belonging to such a system are the elements of two matrices U, Σ, the former of which has n columns and rows, while the latter has n columns and m rows. A column of U with the corresponding column of Σ forms a solution of type (η, μ) of the equations (1). The solutions (u, σ), (v, τ) defined by the equations

$$u = Uc, \quad \sigma = \Sigma c; \qquad v = Ud, \quad \tau = \Sigma d,$$

where c and d are constant multipartite numbers of n elements, satisfy the relation

$$u \cdot \Omega_{v'} - v \cdot \Omega_{u'} = 0, \tag{3}$$

since a similar relation holds for every pair of the solutions of the conjugate system U, Σ. It is important to note that the equation (3) is an identity in the elements of c and d, and would remain true even if these elements were functions of x.

§ 2. *The Proofs of the Theorems in Question.*

Let (η, μ) be an arbitrarily selected set of functions of the type described in the preceding section. If U, Σ is a conjugate system of solutions with determinant $|U|$ different from zero on the interval $x_1 \leqq x \leqq x_2$, the linear equations

$$\eta = Uc \tag{4}$$

determine uniquely the elements of c as functions of x; in other words, there passes a unique solution

$$u = Uc, \quad \sigma = \Sigma c$$

through each point of the curve defined by the functions η in the $(n + 1)$-dimensional $x\eta$-space.

LEMMA. *If the system (η, μ) is a solution of the equations (1), then in every interval $x_1 \leqq x \leqq x_2$ for which the determinant of the matrix U is different from zero*

$$\frac{d}{dx} \eta \cdot (\Omega_{\eta'} - \Omega_{u'}) = R(\eta' - u') \cdot (\eta' - u'). \tag{5}$$

For let (v, τ) be the solution

$$(6) \qquad v = Uc', \quad \tau = \Sigma c'$$

defined at each point of the $x\eta$-curve by the derivatives of the functions c determined by equations (4). With the help of the first of the relations

$$u' = U'c, \quad v' = U'c'$$

it follows from (4) by differentiation that

$$(7) \qquad \eta' = U'c + Uc' = u' + v.$$

Now

$$\frac{d}{dx}\eta\cdot(\Omega_{\eta'} - \Omega_{u'}) = \eta'\cdot(\Omega_{\eta'} - \Omega_{u'}) + \eta\cdot\left(\frac{d}{dx}\Omega_{\eta'} - \frac{\partial}{\partial x}\Omega_{u'} - \Omega_{u'c}\cdot c'\right),$$

where the symbol of partial differentiation indicates that during that operation the elements of c are regarded as constants. The expression $\Omega_{u'}$ is linear in the elements of c, and the expression $\Omega_{u'c}\cdot c'$ is therefore precisely $\Omega_{u'}$ with c replaced by c', that is, $\Omega_{v'}$. The last equation may therefore be written, with the help of equations (1) and the relations $\eta = u$, $\eta' = u' + v$, in the form

$$\begin{aligned}\frac{d}{dx}\eta\cdot(\Omega_{\eta'} - \Omega_{u'}) &= \eta'\cdot(\Omega_{\eta'} - \Omega_{u'}) + \eta\cdot(\Omega_{\eta} - \Omega_{u} - \Omega_{v'}) \\ &= \eta\cdot\Omega_{\eta} + \eta'\cdot\Omega_{\eta'} - u\cdot\Omega_{u} - u'\cdot\Omega_{u'} \\ &\qquad - 2(\eta' - u')\cdot\Omega_{u'} - u\cdot\Omega_{v'} + v\cdot\Omega_{u'}.\end{aligned}$$

But equation (3) holds even when the values c and $d = c'$ are functions of x, and since $\Omega_{\mu} = \Omega_{\sigma} = 0$ the formulas (2) give

$$\frac{d}{dx}\eta\cdot(\Omega_{\eta'} - \Omega_{u'}) = 2\Omega(\eta, \eta', \mu) - 2\Omega(u, u', \sigma) - 2(\eta' - u')\cdot\Omega_{u'}.$$

The expression on the right is readily identified with the second member of (5) by an application of Taylor's formula. For since $\eta = u$

$$\begin{aligned}\Omega(\eta, \eta', \mu) = \Omega(u, u', \sigma) &+ \Omega_{u'}\cdot(\eta' - u') + \Omega_{\sigma}\cdot(\mu - \sigma) \\ &+ \tfrac{1}{2}\Omega_{u'u'}(\eta' - u')\cdot(\eta' - u') + \Omega_{\sigma u'}(\eta' - u')\cdot(\mu - \sigma).\end{aligned}$$

The third term on the right evidently vanishes, and the last also because

$$0 = \Omega_{\mu} - \Omega_{\sigma} = \Omega_{\sigma u'}(\eta' - u').$$

The matrix $\Omega_{u'u'}$ is precisely the matrix R.

Let V, W be two matrices of order n whose columns belong to solutions of equations (1), as described in the preceding paragraphs, without being necessarily conjugate to each other. Let $\Delta(x, \xi)$ represent the determinant of order $2n$

$$\Delta(x, \xi) = \begin{vmatrix} V(x) & W(x) \\ V(\xi) & W(\xi) \end{vmatrix}.$$

The first of the two theorems mentioned is then as follows:

If at every point of an interval $x_1 \leq x \leq x_2$ the quadratic form with matrix R is positive definite, and if furthermore there exists a conjugate system U with determinant $|U| \neq 0$ on the interval, then on the same interval the determinant $\Delta(x, \xi)$ vanishes only at $x = \xi$ or else is identically zero.

Under the hypothesis of the theorem a solution (η, μ) of equations (1) having elements η vanishing simultaneously at two points ξ and ξ' of the interval in question must be identically zero. For the derivative of the function $\eta \cdot (\Omega_{\eta'} - \Omega_{u'})$ expressed by formula (5) is clearly positive or zero between ξ and ξ' since the quadratic form with matrix R is positive definite. If the elements of η vanish at ξ and ξ', so does the function $\eta \cdot (\Omega_{\eta'} - \Omega_{u'})$, and the derivative of this function must be identically zero between ξ and ξ'. In that case the elements of $\eta' - u' = v$ are identically zero, and equations (7) and (6) show that the functions c are constant. The elements of η can not then vanish at all unless the elements of c are all zero, since the determinant of the matrix U in equations (4) is different from zero.

If $\Delta(\xi', \xi) = 0$ for a value $\xi' \neq \xi$, it is always possible to find a set of functions

$$\eta(x) = V(x) \cdot c + W(x) \cdot d$$

linear in the $2n$ constants c and d and vanishing at ξ and ξ'. These functions η belong to solutions of the linear equations (1), and by the reasoning of the preceding paragraph must be identically zero. But in this case it follows readily that $\Delta(x, \xi) \equiv 0$ since c, d are not both zero.

It can be shown that for every point ξ there exists a matrix U of conjugate solutions with determinant different from zero at ξ.* The last theorem then shows that the zero $x = \xi$

* Von Escherich, *Wiener Berichte*, vol. 108 (1899), p. 1339; Bolza, loc. cit., p. 627.

of the determinant $\Delta(x, \xi)$ is an isolated one. This is the second theorem of Bolza mentioned in the introduction above.

The formula (5) is identical with a formula of von Escherich* when the values of c from equations (4) are substituted and η replaced by z. The proof here is, however, of an entirely different character and by far more simple than his. Bolza uses the formula of von Escherich for the purpose of transforming the second variation, and with the help of this transformation deduces the theorem last given. This process seems very much less direct than the argument given above.

UNIVERSITY OF CHICAGO.

CONCERNING A NON-METRICAL PSEUDO-ARCHIMEDEAN AXIOM.

BY DR. ROBERT L. MOORE.

(Read before the American Mathematical Society, April 26, 1913.)

§ 1. *Introduction.*

LET H_1 denote Hilbert's *plane* Axioms of Groups I and II† or Veblen's Axioms I–VIII.‡ Let H_2 denote H_1 together with Desargues' theorem§ (considered as an axiom) and Hilbert's III (axiom of parallels). It is well known that if a two-dimensional space S satisfies H_2 together with Hilbert's congruence axioms of Group IV and the archimedean axiom that of any two non-congruent segments some multiple of the smaller is larger than the greater, then S is either an ordinary euclidean space of two dimensions or an everywhere dense subset of such a space.

Consider the following non-metrical pseudo-archimedean axiom:

AXIOM A. *If* (1) *the points of a line l (Fig. 1) are divided into two sets S_1 and S_2 such that no point of either of these sets is between two points of the other one and such that no point P is*

* Loc. cit., p. 1283, formula (9); Bolza, loc. cit., p. 630, formula (68).

† D. Hilbert, Foundations of Geometry, translated by E. J. Townsend, Open Court Publishing Co., Chicago, 1902.

‡ O. Veblen, "A system of axioms for geometry," *Transactions Amer. Math. Society*, vol. 5 (1904), pp. 343–384.

§ Cf. Hilbert, loc. cit., p. 71.

between every point of S_1 distinct from P and every point of S_2 distinct from P, (2) *A and B are distinct points on the same side of l,* (3) *t is a triangle whose interior contains a point of S_1 and a point of S_2;*

then there exists within t, and on the far side of l from A and B, a point C such that the interior of the triangle ABC contains a point of S_1 and a point of S_2.

In the present paper it will be shown that every two-

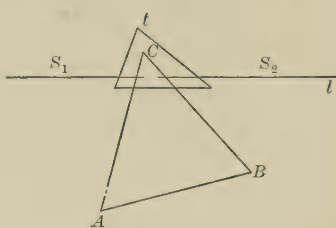


FIG. 1.

dimensional space that satisfies H_1 and Axiom A is equivalent, from the standpoint of analysis situs, either to an ordinary euclidean space of two dimensions or to an everywhere dense subset of such a space. For a precise formulation of this proposition and for a theorem concerning the part that Axiom A plays in connection with the system H_2, the reader is referred to § 3.*

In view of these results it is clear that the non-metrical Axiom A plays a rôle which is, in certain respects, analogous to that played by the above mentioned *metrical* archimedean axiom.

§ 2. *Deductions from H_1 and Axiom A.*

THEOREM 1. *If P is a point on a line l, then there exist on l two countably infinite sequences of points $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3, \cdots$ such that* (1) *for every m and n, P is between A_m and B_n,* (2) *A_{n+1} and B_{n+1} are between A_n and B_n,* (3) *if $P' \neq P$ there exists an n such that the segment A_nB_n does not contain P'.*

Proof. There exist points O, M_1, N, A and C such that O is on l but distinct from P, M_1 is not on l, P is between M_1 and N, A is within the angle NPO and C is between A and

* With regard to another form of non-metrical pseudo-archimedean axiom, see references to Vahlen in §§ 2 and 3 below.

P. The segment AM_1 contains a point A_1 in common with the ray PO. The segments CM_1 and PA_1 have a point A_2 in common. There exists a point M_2 on the segment PM_1 and in the order AA_2M_2. The segments CM_2 and PA_2 have in common a point A_3. Continue this process. In general, for every n the following orders hold: AA_nM_n, $PA_{n+1}A_n$, $CA_{n+1}M_n$. Suppose there exists, on the ray PO, a point X such that no point of the sequence $A_1, A_2, A_3, \cdots$ is between P and X. Let S_1 denote the set of all such points $[X]$ together with all points $[Y]$ such that, for some X, Y is on the ray XP. Let S_2 denote the set of all other points of the line l. With the assistance of Axiom A it can be shown that there exists, within the angle OPM_1, a point Z such that the interior of the triangle ACZ contains at least one point K belonging to S_1 and at least one point A_m of the sequence $A_1, A_2, A_3, \cdots$. It is clear that A_{m+1} is between P and K. Thus the supposition that there exists a point X as described above leads to a contradiction. It follows that, for every point P' on the ray PO, the segment PP' contains a point of the sequence $A_1, A_2, A_3, \cdots$.* Similarly, there exists on the ray PO' (where O' is a point in the order OPO') a sequence of points $B_1, B_2, B_3, \cdots$ such that, for every n, B_{n+1} is between B_n and P and such that if P' is a point on the ray PO' then the segment PP' contains a point of the sequence $B_1, B_2, B_3, \cdots$. It is clear that the sequences $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3, \cdots$ satisfy conditions (1), (2) and (3).

THEOREM 2. *If the points of the line l' are divided into two sets S_1 and S_2 such that no point of either of these sets is between two points of the other one, then there exist two sequences of points $A_1', A_2', A_3', \cdots$ and $B_1', B_2', B_3', \cdots$ such that* (*a*) *every A_n' belongs to S_1 and every B_n' belongs to S_2,* (*b*) *for every n the points A_{n+1}' and B_{n+1}' are between A_n' and B_n',* (*c*) *if C' and D' are distinct points on l', there exists n such that the segment $A_n'B_n'$ does not contain both C' and D'.*

Proof. There exist two points A and B lying on the same side of l' (Fig. 2). Between A and B there is a point P. Let l denote the line AB. By Theorem 1 there exist, on the rays

* In connection with two or three theorems in his paper "Curves in non-metrical analysis situs with an application in the calculus of variations," Lennes makes use of an axiom (which I will call Axiom B) to the effect that P is a limit point of the sequence $A_1, A_2, A_3, \cdots$. Cf. *Amer. Jour. of Mathematics*, vol. 33 (1911), p. 305. Cf. also K. T. Vahlen, Abstrakte Geometrie, Leipzig, 1905, p. 156.

PA and PB respectively, sequences $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3, \cdots$ satisfying conditions (1), (2) and (3). By Axiom A there exists, on the remote side of l' from A, a point O_1 such that the interior of the triangle $A_1O_1B_1$ contains a point A_1' of S_1 and a point B_1' of S_2. Within the triangle $A_1'O_1B_1'$ there is a point O_2 such that the interior of the triangle $A_2O_2B_2$

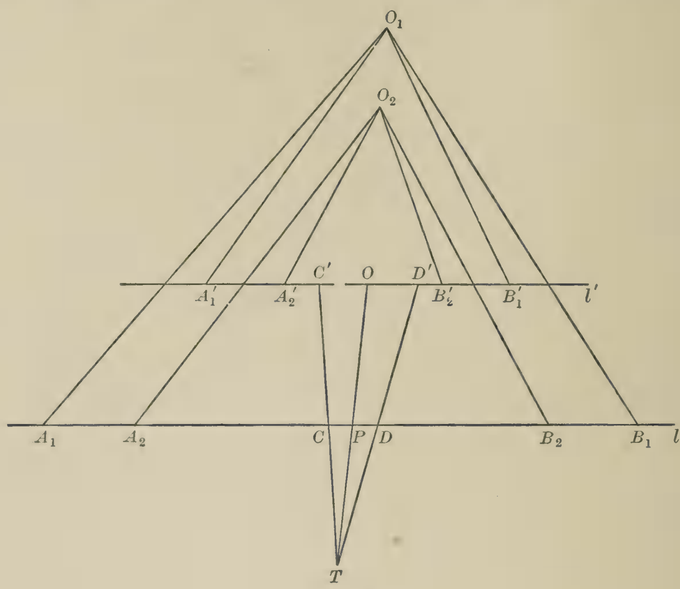


FIG. 2.

contains a point A_2' of S_1 and a point B_2' of S_2 in the order $A_1'A_2'B_2'B_1'$. Continue this process. In general, the point O_{n+1} is within the triangle $A_n'O_nB_n'$, the points A_n' and B_n' belong to S_1 and S_2 respectively and are both within the triangle $A_nO_nB_n$, and the points A_n', A_{n+1}', B_n', B_{n+1}' are in the order $A_n'A_{n+1}'B_{n+1}'B_n'$. It is clear that the sequences $A_1', A_2', A_3', \cdots$ and $B_1', B_2', B_3', \cdots$ satisfy conditions (a) and (b). That they satisfy condition (c) may be proved as follows.

Suppose that they do not satisfy condition (c). Then there exist, on l', two distinct points C' and D' such that for every n the segment $A_n'B_n'$ contains both C' and D' and such that D' is between C' and every B_n'. But between C' and D'

there is a point O. There exists a point T in the order OPT. There exist on the line l points C and D in the orders TCC' and TDD'. There exists m such that A_m and B_m are both between C and D. If neither A_m' nor B_m' is between C' and D' then the lines A_mA_m' and B_mB_m' intersect in a point within or on the triangle TCD. But by hypothesis they intersect on the remote side of l' from A and B. Thus the supposition that condition (c) is not satisfied here leads to a contradiction.

DEFINITION 1. Two segments AB and CD are said to be *separated* if no point or end point of AB is either a point or an end point of CD. Two triangles are said to be *separated* if no point of either of them is on or within the other one.

DEFINITION 2. Suppose $t_1, t_2, t_3, \cdots$ is a countable sequence of triangles such that (1) for every n, t_{n+1} is within t_n, (2) if each of the segments AB and CD intersects every t_n, then AB and CD are not separated. Then the set of all triangles $[t]$ such that the interior of t contains some triangle of the sequence $t_1, t_2, t_3, \cdots$ is called an *ideal point*. The sequence $t_1, t_2, t_3, \cdots$ is said to be a *fundamental sequence* for this ideal point. If α is an ideal point, t_α denotes one of the triangles of which α is composed.

DEFINITION 3. If the line l intersects every triangle of the ideal point α then l is said to *contain* α, and α is said to *lie on* l. If some triangle of α lies on a given side of l then α is said to lie on that side of l.

THEOREM 3. *If α and β are distinct ideal points then there exist two triangles which belong to α and β respectively and are separated from each other.*

THEOREM 4. *If t_α' and t_α'' are triangles belonging to the ideal point α then there exists a triangle t_α''' belonging to α and lying within both t_α' and t_α''.*

THEOREM 5. *If A is a real point, the set of all triangles whose interiors contain A is an ideal point.**

THEOREM 6. *If the points of the line l' are divided into two sets S_1 and S_2 such that no point of either of these sets is between two points of the other one then there exists an ideal point α such that every triangle of α contains a point of S_1 and a point of S_2.*

Proof. There exist, in S_1 and S_2 respectively, sequences

* The ideal point which is determined in this way by the real point A will be denoted by the symbol A^*.

$A_1', A_2', A_3', \cdots$ and $B_1', B_2', B_3', \cdots$ satisfying conditions (*a*), (*b*) and (*c*) of Theorem 2. There exist points P and $\bar{P}$ lying on opposite sides of l'. It follows with the help of Theorem 1 that there exist on the segments PA_1' and PB_1' respectively sequences of points $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3, \cdots$ in the orders $PA_1A_2\cdots A_1'$ and $PB_1B_2\cdots B_1'$ and such that if X is on the ray $A_1'P$ and Y is on the ray $B_1'P$ then there exists m such that A_m is between X and A_1' while B_m is between Y and B_1'. Similarly there exist, on the segments $\bar{P}A_1'$ and $\bar{P}B_1'$ respectively, sequences $\bar{A}_1, \bar{A}_2, \bar{A}_3, \cdots$ and $\bar{B}_1, \bar{B}_2, \bar{B}_3, \cdots$ in the orders $\bar{P}\bar{A}_1\bar{A}_2\cdots A_1'$ and $\bar{P}\bar{B}_1\bar{B}_2\cdots B_1'$ and such that if X is on the ray $A_1'\bar{P}$ and Y is on the ray $B_1'\bar{P}$ then there exists m such that $\bar{A}_m$ is between X and A_1' while $\bar{B}_m$ is between Y and B_1'. For each n the segments $A_n'B_n$ and $B_n'A_n$ have a point O_n in common. There exist points C_n and D_n in the orders $O_nA_{n+1}'C_n$, $A_n'C_n\bar{B}_n$, $O_nB_{n+1}'D_n$, $B_n'D_n\bar{A}_n$. Let t_n denote the triangle $O_nC_nD_n$. Let α denote the set of all triangles [t] such that, for some n, t_n is within t. It can be proved that α is an ideal point. Every triangle of α contains a point of S_1 and a point of S_2.

THEOREM 7. *If the real points A and B and the ideal point α all lie on the same side of the line l and α is not on the line AB then there exists a point C, on the same side of l as A, such that the interior of the triangle ABC contains some triangle of α.*

Proof. Suppose there exists no such point C. If t is any triangle of α which has no point in common with the line AB, there exist lines a_t and b_t containing A and B respectively such that (1) the interior of t is entirely on the A-side of b_t and entirely on the B-side of a_t, (2) the perimeter of t contains a point of a_t and also a point of b_t. Let $\bar{t}$ denote a definite triangle of α whose perimeter has no point in common with either l or AB. Let $t_1, t_2, t_3, \cdots$ be a fundamental sequence of triangles of α such that t_1 lies within $\bar{t}$. For every triangle t that lies within t_1, a_t intersects the perimeter of t_1 in two points A_t and A_t' in the order $AA_t'A_t$ while b_t intersects the perimeter of t_1 in points B_t and B_t' in the order $BB_t'B_t$. Let $A_tY_tB_t$ denote that arc* of the perimeter of t_1 whose end points are A_t and B_t and which contains neither A_t' nor B_t'. Then clearly, for every n, $A_{t_n}Y_{t_n}B_{t_n}$ contains $A_{t_{n+1}}Y_{t_{n+1}}B_{t_{n+1}}$. There are two cases to be considered.

* This arc is either a segment or a broken line.

Case I. Suppose there exist two points $\bar{A}$ and $\bar{B}$ such that on $A_{t_2}Y_{t_2}B_{t_2}$ the order $A_{t_2}A_{t_n}\bar{A}\bar{B}B_{t_n}B_{t_2}$ holds true for every n. Then each of the two segments $A\bar{A}$ and $B\bar{B}$ intersects every t_n. Thus a contradiction is reached.

Case II. Suppose there do not exist two such points $\bar{A}$ and $\bar{B}$. Then it may be easily proved with the help of Axiom A that there exists, within the triangle $\bar{t}$, a point C such that AC and BC intersect t_1 in points A' and B' respectively such that, for some n, the order $A_{t_2}A'A_{t_n}B_{t_n}B'B_{t_2}$ holds true and thus t_n lies within the triangle ABC. Thus again a contradiction is obtained.

THEOREM 8. *If α and β are two distinct ideal points which do not both lie on the line l then there do not exist on l two distinct points A and B such that for every t_α, t_β and point P of the segment AB there is a line through P that intersects t_α and t_β.*

Proof. Suppose there exists such a segment AB. Suppose α is not on l (Fig. 3). There exist two separated triangles $\bar{t}_\alpha$

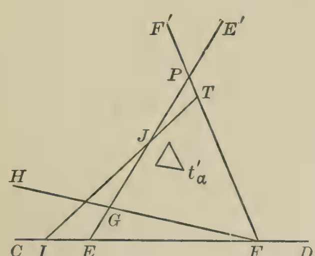


FIG. 3.

and $\bar{t}_\beta$ belonging to α and β respectively. There exists a line $\bar{l}$ distinct from l such that $\bar{t}_\alpha$ and $\bar{t}_\beta$ lie on different sides of $\bar{l}$. There exist, on the segment AB, points C and D such that $\bar{l}$ contains neither C nor D nor any point between C and D. There exist E and F in the order $CEFD$. Suppose* $\bar{t}_\alpha$ is on the same side of $\bar{l}$ as E. By Theorem 7 there exists a point P on the E-side of $\bar{l}$ and a triangle t_α' belonging to α such that t_α' lies entirely within the triangle EPF. Clearly $\bar{t}_\beta$ is entirely without the triangle EPF. There exist points

*Every other case may be either easily disposed of or reduced to a case which is, apart from notation, the same as this one.

E' and F' in the orders EPE' and FPF'. If every triangle of β contained a point within the angle EPF' and also a point within the angle FPE' then each of the lines EP and FP would intersect every triangle of β. But not every triangle of β contains P. Thus β would not be an ideal point according to Definition 2. It follows that one of the angles EPF' and FPE' is such that some triangle of β contains no point within or on that angle. Suppose this is true of the angle FPE'. Then, by Theorem 4, there exists a triangle belonging to β and containing no point within or on the triangle EPF or within or on the angle FPE'. There exists, between E and P, a po nt G such that no point of t_α' is within the triangle FEG. But for every t_β there is a line passing through F and intersecting t_α' and t_β. It follows that every t_β contains a point within the angle HGP where H is a point in the order FGH. Hence there exists a triangle t_β' belonging to β and lying entirely within the angle CEP. There exist points I, J and T in the orders CIE, EJP, FTP and IJT and such that t_α' is within the triangle ITF while t_β' is within the angle IJP. But I is collinear with a point of t_β' and a point of t_α'. It follows that t_β' contains a point within the triangle IJE. But t_β' is entirely within the angle IJP. Thus the supposition that Theorem 8 is false leads to a contradiction.

Definition 4. If α and β are distinct ideal points, the *ideal line* $\alpha\beta$ is the set of all ideal points $[\gamma]$ such that for every t_α, t_β, t_γ there exist three collinear real points A, B, C such that A is within t_α, B is within t_β and C is within t_γ. If α, β and γ are distinct ideal points such that for every t_α, t_β, t_γ there exist real points A, B and C within t_α, t_β and t_γ respectively and in the order ABC then α β and γ are said to be *in the order* $\alpha\beta\gamma$.

Theorem 9. *If A and B are two distinct real points and α and β are two distinct ideal points such that neither A^* nor B^* is on the ideal line $\alpha\beta$, and if, furthermore, for every two triang es t_α and t_β, belonging to α and β respectively, there exists, between A and B a point which is collinear with some point of t_α and some point of t_β then there exist t_α and t_β such that every point of l which is collinear with a point of t_α and a point of t_β lies be- ween A and B and there exists, on the ideal line $\alpha\beta$, an ideal point γ in the order $A^*\gamma B^*$.*

Theorem 10. *If α and β are distinct ideal points, there exists an ideal point γ in the order $\alpha\beta\gamma$.*

THEOREM 11. *If* α, β *and* γ *are three ideal points in the order* $\alpha\beta\gamma$ *and* $\bar{t}_\alpha$, $\bar{t}_\beta$, $\bar{t}_\gamma$ *are mutually separated triangles belonging to* α, β *and* γ *respectively, then there exists* t_β *such that* (1) *every point of* t_β *is collinear with some point of* $\bar{t}_\alpha$ *and some point of* $\bar{t}_\gamma$, (2) *if* P_α, P_β, P_γ *are collinear points belonging to* $\bar{t}_\alpha$, t_β *and* $\bar{t}_\gamma$ *respectively then* P_α, P_β, P_γ *are in the order* $P_\alpha P_\beta P_\gamma$.

THEOREM 12. *If the ideal points* α, β, γ *are in the order* $\alpha\beta\gamma$ *then they are not in the order* $\beta\gamma\alpha$.

THEOREM 13. *If the ideal point* β *lies on the ideal line* $\alpha\gamma$ *then for every* t_β *there exist* t_α *and* t_γ *such that every line that intersects* t_α *and* t_γ *intersects also* t_β.

Proof. There exists, within t_β, a triangle t_β' belonging to β. For every t_α and t_γ there is a line intersecting t_α, t_γ and t_β'. It follows with the aid of Theorem 9 that, for some lettering A, B, C of the vertices of t_β', it is true that, for every point X in the order ABX there exist triangles $t_{\alpha X}$ and $t_{\gamma X}$ belonging to α and γ respectively such that every line that intersects $t_{\alpha X}$ and $t_{\gamma X}$ intersects also the segment AX. But X may be chosen within the triangle t_β. Thus there exist t_α and t_γ such that every line that intersects both t_α and t_γ intersects also t_β.

THEOREM 14. If γ_1, γ_2, γ_3 *are three distinct ideal points lying on the ideal line* $\alpha\beta$ *then* γ_1 *is on the ideal line* $\gamma_2\gamma_3$.

Proof. Suppose t_{γ_1}, t_{γ_2} and t_{γ_3} are triangles belonging to γ_1, γ_2 and γ_3 respectively. By Theorem 13 there exist triangles $t_\alpha^{(1)}$, $t_\alpha^{(2)}$, $t_\alpha^{(3)}$, belonging to α, and triangles $t_\beta^{(1)}$, $t_\beta^{(2)}$, $t_\beta^{(3)}$, belonging to β, such that if $i = 1, 2$ or 3 then every line that intersects $t_\alpha^{(i)}$ and $t_\beta^{(i)}$ intersects also t_{γ_i}. But by Theorem 4 there exist $\bar{t}_\alpha$ and $\bar{t}_\beta$ belonging to α and β respectively such that $\bar{t}_\alpha$ is within every $t_\alpha^{(i)}$ and $\bar{t}_\beta$ is within every $t_\beta^{(i)}$ $(i = 1, 2, 3)$. Every line that intersects $\bar{t}_\alpha$ and $\bar{t}_\beta$ must intersect t_{γ_1}, t_{γ_2} and t_{γ_3}. It follows that γ_1 is on $\gamma_2\gamma_3$.

THEOREM 15. *If* α, β, γ *and* δ *are four distinct ideal points no three of which are collinear* and for every* t_α, t_β, t_γ, t_δ *there exists a point which is collinear with a point of* t_α *and a point of* t_β *and which at the same time is between a point of* t_γ *and a point of* t_δ, *then there exists, on the ideal line* $\alpha\beta$, *an ideal point* ϵ *in the order* $\gamma\epsilon\delta$.

Proof. Since no three of the points α, β, γ and δ are

* Three or more ideal points are said to be collinear if there exists an ideal line which contains them all.

collinear there exist four mutually separated triangles $\bar{t}_\alpha, \bar{t}_\beta, \bar{t}_\gamma, \bar{t}_\delta$ belonging to $\alpha, \beta, \gamma, \delta$ respectively and such that no line intersecting $\bar{t}_\alpha$ and $\bar{t}_\beta$ intersects either $\bar{t}_\gamma$ or $\bar{t}_\delta$. Let $t_{\alpha 1}, t_{\alpha 2}, t_{\alpha 3}, \cdots, t_{\beta 1}, t_{\beta 2}, t_{\beta 3}, \cdots, t_{\gamma 1}, t_{\gamma 2}, t_{\gamma 3}, \cdots, t_{\delta 1}, t_{\delta 2}, t_{\delta 3}, \cdots$ be fundamental sequences belonging to $\alpha, \beta, \gamma, \delta$ respectively and such that $t_{\alpha 1}, t_{\beta 1}, t_{\gamma 1}, t_{\delta 1}$ lie within $\bar{t}_\alpha, \bar{t}_\beta, \bar{t}_\gamma, \bar{t}_\delta$ respectively. For each positive integer n there exists a convex polygon p_n, of six sides or less, such that (1) with the exception of two sides, every side of p_n is a side of $t_{\gamma n}$ or a side of $t_{\delta n}$, (2) each of the two remaining sides of p_n has for one end point a point of $t_{\gamma n}$ and for its other end point a point of $t_{\delta n}$, (3) the interior of p_n contains the interiors of $t_{\gamma n}$ and $t_{\delta n}$. By hypothesis there exists, within p_n, a point X_n which is collinear with a point of $t_{\alpha n}$ and a point of $t_{\beta n}$. It follows that there exists a quadrilateral $A_nB_nC_nD_n$ such that (1) every point on or within $A_nB_nC_nD_n$ is between a point of $t_{\gamma n}$ and a point of $t_{\delta n}$ and is also on some line that intersects $t_{\alpha n}$ and $t_{\beta n}$, (2) every point which is common to a line intersecting $t_{\alpha n}$ and $t_{\beta n}$ and a line intersecting $t_{\gamma n}$ and $t_{\delta n}$ is on or within $A_nB_nC_nD_n$. If there exists m such that for every n greater than m the interior of $A_nB_nC_nD_n$ contains a point of the diagonal A_mC_m then it follows with the help of Theorem 9 that there exists, on A_mC_m, an ideal point which is collinear with α and β and, at the same time, is between γ and δ. If there exists no such m then the sequence of triangles $A_1B_1C_1$, $A_1C_1D_1$, $A_2B_2C_2$, $A_2C_2D_2, \cdots$, contains, as a subsequence, an infinite sequence $t_1, t_2, t_3, \cdots$ such that, for every n, t_{n+1} is within t_n. There do not exist two separated segments each of which contains a point of every t_n. For if there did exist two such segments, s_1 and s_2, they would* contain two ideal points ϵ_1 and ϵ_2 respectively such that ϵ_1 and ϵ_2 are both on $\alpha\beta$ and both on $\gamma\delta$ and therefore, by Theorem 14, α, β, γ and δ would be collinear, which is contrary to hypothesis. Hence if ϵ denotes the set of all triangles $[t]$ such that the interior of t contains at least one triangle of the sequence $t_1, t_2, t_3, \cdots$ then ϵ is an ideal point. It is clear that ϵ is between γ and δ and collinear with α and β.

THEOREM 16. *If α, β and γ are non-collinear ideal points and δ and ϵ are two ideal points in the orders $\alpha\beta\delta$ and $\beta\epsilon\gamma$ then there exists, on the ideal line $\delta\epsilon$, an ideal point η in the order $\gamma\eta\alpha$.*

Proof. There exist mutually separated triangles $t_{\alpha 1}, t_{\beta 1}$,

* Cf. Theorem 9.

$t_{\gamma 1}$, $t_{\delta 1}$, $t_{\epsilon 1}$, belonging to α, β, γ, δ, ϵ respectively and such that no point of $t_{\gamma 1}$ is on a line intersecting $t_{\alpha 1}$ and $t_{\beta 1}$. If $t_{\alpha 2}$ and $t_{\delta 2}$ are triangles of α and δ respectively lying within $t_{\alpha 1}$ and $t_{\delta 1}$ respectively then, by Theorem 11, there exists, within $t_{\beta 1}$, a triangle $t_{\beta 2}$ belonging to β and such that every point of $t_{\beta 2}$ is between some point of $t_{\alpha 2}$ and some point of $t_{\delta 2}$. Let $t_{\gamma 2}$ and $t_{\epsilon 2}$ denote triangles belonging to γ and ϵ respectively. By hypothesis there exist within $t_{\beta 2}$, $t_{\epsilon 2}$ and $t_{\gamma 2}$, three points, P_β, P_ϵ and P_γ respectively, in the order $P_\beta P_\epsilon P_\gamma$. There exist points P_α and P_δ within $t_{\alpha 2}$ and $t_{\delta 2}$ respectively and in the order $P_\alpha P_\beta P_\delta$. Since P_α, P_β and P_γ are not collinear and furthermore P_β is between P_α and P_δ while P_ϵ is between P_β and P_γ, it follows that there exists a point P_η in the orders $P_\delta P_\epsilon P_\eta$ and $P_\gamma P_\eta P_\alpha$. It follows by Theorem 15 that there exists, on the ideal line $\delta\epsilon$, an ideal point η in the order $\gamma\eta\alpha$.

THEOREM 17. *If the points of an ideal line are divided into two sets S_1 and S_2 such that no point of either of these sets is between two points of the other one, then there exists an ideal point α which lies between every point of S_1 distinct from α and every point of S_2 distinct from α.*

THEOREM 18. *The set of all ideal points satisfies Veblen's Axioms I–VIII, XI (also Hilbert's plane axioms of Groups I and II together with the Dedekind cut postulate).*

§ 3. *Conclusion.*

DEFINITION. A space S consisting of a definite system of *points* and *lines* subject to definite relations of *alignment* and *order* is said to be descriptively equivalent to a subset S' of an ordinary euclidean space E if there exists between the *points* of S and the points of S' a one-to-one reciprocal correspondence preserving collinearity and order.*

DEFINITION. A two-dimensional space S consisting of a definite system of *points* and *lines* subject to definite relations of *alignment* and *order* is said to be equivalent, from the standpoint of analysis situs, to a subset S' of a two-dimensional euclidean space E if there exists, between the *points* of S

* The statement that such a correspondence preserves collinearity and order signifies that if A, B, C are three *points* of S and A', B', C' respectively are the corresponding points of S' then A, B, C are in the *order* ABC on a line in S if and only if A', B', C' are in the order $A'B'C'$ on a line in E.

and the points of S', a one-to-one reciprocal correspondence preserving limits.*

The following theorems may be easily established with the assistance of Theorem 18 of § 2 and Theorem IV of my paper "On a set of postulates which suffice to define a number-plane."†

THEOREM A. *Every two-dimensional space that satisfies Hilbert's plane axioms of Groups I and II (or Veblen's I–VIII) together with Axiom A is equivalent, from the standpoint of analysis situs, either to a two-dimensional euclidean space or to an everywhere dense subset thereof.*

THEOREM B.‡ *Every two-dimensional space that satisfies Hilbert's plane axioms of Groups I, II and III (or Veblen's I–VIII, XII) together with Desargues' theorem and Axiom A is descriptively equivalent either to a two-dimensional euclidean space or an everywhere dense subset thereof.*

COROLLARY. *Pascal's theorem§ is a consequence of Hilbert's plane axioms of Groups I, II and III together with Desargues' theorem and Axiom A.*

THE UNIVERSITY OF PENNSYLVANIA.

A TYPE OF SINGULAR POINTS FOR A TRANSFORMATION OF THREE VARIABLES.

BY DR. W. V. LOVITT.

(Read before the American Mathematical Society, December 31, 1915.)

IN the *Transactions* for October, 1915, I discussed some singularities of a point transformation in three variables

$$(1) \qquad x = \phi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w)$$

* The statement that such a correspondence preserves limits signifies that if A is a *point* of S, M is a *point set* of S, and A' and M' respectively are the corresponding point and point set of S' then P is a *limit point* of M if, and only if, P' is a limit point of M'. Here P is said to be a *limit point* of M if, and only if, every *triangle* of S that contains P *within* it contains *within* it at least one *point* of M distinct from P.

† *Transactions of the American Mathematical Society*, vol. 16 (1915), pp. 27–32.

‡ For a corresponding theorem regarding Axiom B (cf. footnote in § 2) see Vahlen, loc. cit., pp. 158–163.

§ Cf. Hilbert, loc. cit., p. 40.

with determinant

$$J(u, v, w) = \begin{vmatrix} \phi_u & \phi_v & \phi_w \\ \psi_u & \psi_v & \psi_w \\ \chi_u & \chi_v & \chi_w \end{vmatrix}.$$

In that paper the functions ϕ, ψ, χ were not necessarily analytic but it was presupposed that

(*a*) the functions ϕ, ψ, χ are of class C'''* in a neighborhood of the origin $(u, v, w) = (0, 0, 0)$;

(*b*) the following initial conditions are satisfied:

$$\phi(0, 0, 0) = \psi(0, 0, 0) = \chi(0, 0, 0) = 0;$$

(*c*) $J(0, 0, 0) = 0$;

(*d*) at the origin $(u, v, w) = (0, 0, 0)$ at least one of the determinants of the matrix

$$\begin{matrix} J_u & J_v & J_w \\ \phi_u & \phi_v & \phi_w \\ \psi_u & \psi_v & \psi_w \\ \chi_u & \chi_v & \chi_w \end{matrix}$$

is different from zero.

In the present note I desire to show that the results of that paper apply to a transformation of the form

$$(2) \qquad \begin{aligned} f(x, y, z;\ u, v, w) &= 0, \\ g(x, y, z;\ u, v, w) &= 0, \\ h(x, y, z;\ u, v, w) &= 0. \end{aligned}$$

The functions f, g, h are not necessarily analytic but it will be presupposed that

(a') the functions f, g, h are of class C''' in a neighborhood of the origin $(x, y, z;\ u, v, w) = (0, 0, 0;\ 0, 0, 0)$;

(b') the following initial conditions are satisfied:

$$f(0, 0, 0; 0, 0, 0) = g(0, 0, 0; 0, 0, 0) = h(0, 0, 0; 0, 0, 0) = 0;$$

(c') $B \equiv \dfrac{\partial(f, g, h)}{\partial(u, v, w)} = 0$, at the origin;

(d') at the origin $(x, y, z;\ u, v, w) = (0, 0, 0;\ 0, 0, 0)$ at least one of the determinants of the matrix

* We shall say that a single-valued function f of (u, v, w) is of class C''' if $f(u, v, w)$ and its partial derivatives of orders one, two, and three are continuous in a region in which f is defined.

$$\begin{matrix} B_u & B_v & B_w \\ f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{matrix}$$

is different from zero;

(e') $\dfrac{\partial(f, g, h)}{\partial(x, y, z)} \neq 0$, at the origin.

Assumption (e') assures us of the existence of a solution of equations (2) of the form of equations (1). We shall now consider that equations (1) have been obtained from (2), and proceed to show that, on account of the conditions (a'), (b'), (c'), (d'), (e'), equations (1) satisfy the conditions (a), (b), (c), (d).

Assumptions (a) and (b) follow at once from (a') and (b') as a result of the ordinary theorems on implicit functions.* From the equation

$$\frac{\partial(f, g, h)}{\partial(x, y, z)} \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{\partial(f, g, h)}{\partial(u, v, w)}, \tag{3}$$

on account of our assumptions (c') and (e'), we find

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = 0, \text{ at the origin.}$$

The left hand member of the last equation is the jacobian J of the equations (1) and hence the condition (c) is satisfied. From equation (3) it follows, on account of (e') and (c), that if not all of B_u, B_v, B_w, then also not all of J_u, J_v, J_w, vanish at the point in question. It is easily verified that if the determinant B is of rank two, then also is the determinant J of rank two. Hence it follows from assumption (d') that the assumption (d) is satisfied.

That the determinant J is of rank two is seen as follows. Let α, β stand for some two of the variables u, v, w. Then

$$f_x\phi_\alpha + f_y\psi_\alpha + f_z\chi_\alpha = - f_\alpha,$$

$$f_x\phi_\beta + f_y\psi_\beta + f_z\chi_\beta = - f_\beta,$$

$$g_x\phi_\alpha + g_y\psi_\alpha + g_z\chi_\alpha = - g_\alpha,$$

$$g_x\phi_\beta + g_y\psi_\beta + g_z\chi_\beta = - g_\beta.$$

* See Bliss, Princeton Colloquium Lectures, p. 8.

Whence

$$f_\alpha g_\beta - f_\beta g_\alpha = \frac{\partial(\phi\psi)}{\partial(\alpha\beta)} f_x g_y + \frac{\partial(\phi\chi)}{\partial(\alpha\beta)} f_x g_z + \frac{\partial(\psi\phi)}{\partial(\alpha\beta)} f_y g_x + \frac{\partial(\psi\chi)}{\partial(\alpha\beta)} f_y g_z.$$

Now if the determinant B is of rank two we must have

$$f_\alpha g_\beta - f_\beta g_\alpha \neq 0, \text{ at the origin;}$$

whence it follows that some one of the functional determinants appearing on the right hand side of the last equality does not vanish at the origin, and hence J is of rank two.

PURDUE UNIVERSITY.

THE HISTORY OF THE CONSTRUCTION OF THE REGULAR POLYGON OF SEVENTEEN SIDES.

Die elementargeometrischen Konstruktionen des regelmässigen Siebzehnecks. Eine historisch-kritische Darstellung. Von R. GOLDENRING. Leipzig und Berlin, Teubner, 1915. 8vo. 6+69 pp. Price 2.80 marks.

TILL near the close of the eighteenth century, mathematicians felt sure that the only regular polygons which could be constructed with ruler and compasses were those known to the Greeks. But the extraordinary discoveries of Gauss, while yet in his teens, greatly extended this class of polygons and settled for all time the limits of possibilities for such constructions. In this connection the discovery that the regular polygon of seventeen sides could be constructed with ruler and compasses was not only one of which Gauss was vastly proud throughout his life, but also, according to Sartorius von Waltershausen,* one which decided him to dedicate his life to the study of mathematics. Two of Gauss's notes recording this turning point of his career have been preserved. The very first entry in his "Wissenschaftliches Tagebuch 1796–1814" † is: "Principia quibus innititur sectio circuli ac divisibilitas eiusdem geometrica in septemdecim partes etc. *Mart.* 30. *Brunsvigae.*" And again, in his own copy of his Disquisitiones

* Gauss zum Gedächtniss, Leipzig, 1856, p. 16.

† This was "mit Anmerkungen herausgegeben von F. Klein," *Math. Annalen*, Band 57 (1903), pp. 1–34.

Arithmeticae* he wrote the following note† in the margin beside article 365: "Circulum in 17 partes divisibilem esse geometrice, deteximus 1796 Mart. 30."

The first published announcement of this discovery occurred about two months later in the *Intelligenzblatt* of the famous *Allgemeine Literatur-Zeitung.*‡ As files of this periodical are rare§ on this side of the Atlantic, it would seem to be worth while to make more accessible an exact transcription of the announcement. It is as follows:—

"III. Neue Entdeckungen.

Es ist jedem Anfänger der Geometrie bekannt, dass verschiedene ordentliche Vielecke, namentlich das Dreyeck, Viereck, Funfzehneck, und die, welche durch wiederholte Verdoppelung der Seitenzahl eines derselben entstehen, sich geometrisch construiren lassen. So weit war man schon zu Euklids Zeit, und es scheint, man habe sich seitdem allgemein überredet, dass das Gebiet der Elementargeometrie sich nicht weiter erstrecke: wenigstens kenne ich keinen geglückten Versuch, ihre Grenzen auf dieser Seite zu erweiten.

Desto mehr, dünkt mich, verdient die Entdeckung Aufmerksamkeit, dass *ausser jenen ordentlichen Vielecken noch eine Menge anderer, z. B. das Siebenzehneck, einer geometrischen Construction fähig ist.* Diese Entdeckung ist eigentlich nur ein Corollarium einer noch nicht ganz vollendeten Theorie von grösserem Umfange, und sie soll, sobald diese ihre Vollendung erhalten hat, dem Publicum vorgelegt werden.

C. F. Gauss, a. Braunschweig,
Stud. der Mathematik zu Göttingen.

Es verdient angemerkt zu werden, dass Hr. Gauss jetzt in seinem 18ten Jahre steht, und sich hier in Braunschweig mit eben so glücklichem Erfolg der Philosophie und der classischen Litteratur als der höheren Mathematik gewidmet hat.

Den 18 April 96.

E. A. W. Zimmermann, Prof."

Five years later Gauss published the "Theorie von grösserem Umfange" in his Disquisitiones Arithmeticae. The only other reference to the regular polygon of seventeen sides, in

---

* Lipsiæ, 1801; also in Werke, Bd. 1, Göttingen, 1870.

† *Cf.* Werke, Band 1, p. 476.

‡ Nr. 66, 1 Junius, 1796, col. 554. Two mistakes in this reference are made by Klein, l. c.

§ There is a set in the library of Columbia University.

Gauss's Werke, is in connection with a report of a paper delivered by Erchinger before the Royal Society of Göttingen in 1825.* Gauss gives Erchinger's geometrical construction of the regular 17-side† and remarks that it flows naturally from equations which he had given in the Disquisitiones. He then points out that the merit of Erchinger's paper was not so much in this construction as in the synthetic "proof‡ of its correctness and this is carried through with such admirable, painstaking care to avoid anything not elementary, that it reflects honor on the author and inspires the hope, that his truly uncommon mathematical talent may find every encouragement." While Gauss refers to two of the earlier synthetic constructions of Paucker,§ he remarks that that of Erchinger is "different and carried through more in the spirit of pure geometry."

Goldenring commences his little work by showing that the problem of the solution of the equation $x^{17} = 1$ may be reduced to the solution of certain quadratic equations. Geometrical solutions of such equations are indicated in the next nine pages. Then follow about a score of geometrical constructions for the regular polygon of seventeen sides. They include, of course, the geometrographical construction of Güntsche (1902), the so-called Steinerian constructions of von Staudt (1842) and Schröter (1872), and the Mascheronian construction of Gérard (1897). Two of the constructions given are claimed as new: one by Professor Haussner, of the University of Jena, to whom the book is dedicated, and one by the author himself, through inversion of the von Staudt-Schröter figure. In an appendix (pages 65–66) is given, without any indication of authorship, a construction by means of a right angle.‖

* "Geometrische Construction des regelmässigen Siebenzehnecks," *Goettingische Gelehrte Anzeigen*, Dec. 19, 1825, no. 203, p. 2025; Werke, Band 2, pp. 186–187. See also *Bulletin des Sciences mathématiques*, vol. 5 (1826), pp. 299–300.

† Curiously enough, Goldenring gives the impression (pages 15 and 68) that Erchinger's *construction* is unknown.

‡ Unfortunately this proof has not been preserved.

§ (1) "Geometrische Verzeichnung des regelmässigen 17-Ecks und 257-Ecks in d. Kreis," *Jahresverhandl. d. kurländische Gesellschaft für Literatur und Kunst*, Mitau, Band 2, 1822. Apparently unknown to Goldenring. (2) Die ebene Geometrie der geraden Linie und des Kreises. Königsberg, 1823, p. 187. Paucker is also the author of: (3) De divisione geometrica peripheriæ circuli in XVII partes æquales, Königsberg, 1817. This date is incorrectly given as 1814 by Goldenring.

‖ This is due to Adler, Theorie der geometrischen Konstruktionen,

Until the publication of Goldenring's pamphlet the most elaborate account of the work done in connection with constructions of the regular 17-side was in the sketch by E. Daniele, "Sulle costruzioni dell' ettadecagono regolare."* While Goldenring has performed a service in presenting something much more elaborate, which is also usefully arranged, it is far from being anything like complete. The only reference to Gauss is to his Disquisitiones and at least two geometrical solutions, published several years before Erchinger's, are nowhere mentioned. These are by John Lowry (1819)† and Samuel Jones (1820).‡ Again, we are informed (page 67) that Ampère's construction announced to the French Academy in 1835§ "does not appear to have been published"; but surely this is the solution "attribuée à Ampère," published, since 1844, in at least five editions of La Frémoire and Catalan's "Théorèmes et problèmes."|| This solution was also given

---

Leipzig, 1906, p. 227. See also A. Mitzscherling, Das Problem der Kreisteilung, Leipzig, 1913, pp. 73–74.

* Questioni riguardanti le matematiche elementari raccolte e coordinate da F. Enriques. Vol. 2, Bologna, 1914, pp. 167–183.

† *The Mathematical Repository*, new series, vol. 4 (1819), p. 160. Lowry's proof occupies pages 160–168.

‡ The paper dated "Dublin, 17th October, 1819" and read Jan. 24, 1820, was published in *Transactions of the Irish Academy*, vol. 13 (1818), pp. 175–187.

§ "Division de la circonférence de cercle," *Comptes rendus de l'Acad. d. Sc.*, vol. 1 (1835), pp. 119–120. It is here stated that M. Ampère had presented to the academy a geometric figure in which was represented a very simple construction for dividing the circumference of a circle into 17 equal parts. He also announced that he soon expected to read a note the "aim of which was to make clear to those who are still studying elementary geometry, why one can divide, with ruler and compasses, the circumference of a circle into a prime number of equal parts only when this number exceeds unity by a power of 2. M. Ampère will indicate, at the same time, a method leading to this end, that is to say to the desired division in all possible cases and that, without having recourse to any of the theories of higher algebra."

"The utility of introducing these notions into treatises of elementary geometry and the question of why the ancients did not discover the division into 17 parts were questions which led to a discussion by MM. Poinsot, Ampère and Libri. We abstain from giving this discussion at the present time since it has been indicated that it will be renewed at the time when M. Ampère will read the note which he has been content to simply announce to-day."

Apparently this new note was never read. Ampère died in the following year.

|| Théorèmes et problèmes de géométrie élémentaire par H. C. de la Frémoire. Second édition entièrement revue et corrigée par E. Catalan. Paris, 1852, pp. 178–180, 207–209. Sixième éd. 1879, pp. 267–269, 298–302; La Frémoire's name no longer appears on the title page of this edition.

in abridged form by John Casey.*

In the Bibliography (pages 67–69) there are references to 7 pamphlets and 17 special articles on the regular polygon of seventeen sides, and there are some 15 references to more general articles or books in which the same topic is treated. My copy of H. A. Rothe's pamphlet De divisione peripheriæ circuli in 17 et 13 partes æquales was published at Erlangen in 1805 not 1804 (page 67).† There are many omissions in the Bibliography. Some of these have been already indicated above. Here are more references which I happen to have met with in the last few years.‡

J. J. Barniville, *Educ. Times Repr.*, vol. 54 (1891), p. 28, question 10176—Bochow, "Eine einfache Berechnung des 17 Ecks," *Zeitschrift für Math. u. Phys.* (Schlömilch), vol. 38 (1893), pp. 250–252—A. Cayley, "On the equation $x^{17}-1=0$," *Messenger of Math.*, vol. 19 (1890) pp. 184–188; Collected Papers, vol. 13 (1897) pp. 60–63—C. H. Chepmell, "In a given circle to inscribe the regular polygon of thirty-four sides," *Educ. Times*, March, 1911; *Educ. Times Repr.* (2), vol. 20 (1911), pp. 51–54—E. Collignon, "Construction du polygone régulier de 17 côtés," *Ass. Franc. Comptes R.*, tome 8 (1879), pp. 162–169—L. Gérard, "Construction du polygone régulier de 17 côtés," *Bull. de Math. élêmentaires*, tome 2 (mars, 1897), pp. 164–167—J. A. Grunert, "Reguläre Sieb-

The Kauffmann-Reuschle German translation (Stuttgart, 1858) of the second edition contains (p. 155) the definite statement: "Der hier angeführte geometrische Beweis ist von Ampère."

* For example in his Elements of Euclid, 16th ed., Dublin, 1897, pp. 302–304.

† While there seems to be authority for the statement (p. 67) that H. Birnbaum's paper, "Ueber das reguläre Siebzehneck," was published as a pamphlet in 1834 (e. g., L. A. Sohncke's Bibliotheca Mathematica, Leipzig, 1854, p. 140), there was an edition in 1833 in connection with a school programme, the title page of which contains the invitation: "Zu der öffentlichen Freitags den 29 März 1833 zu haltenden Prüfung der drei obern Classen des Helmstedt-Schöningenschen Gymnasiums und zu dem damit verbundenen Redeacte ladet die Eltern so wie alle Freunde des Schulunterrichts mit geziemender Ehrerbietung ein Dr. Philipp Karl Hess, Professor und Director."

‡ The most extensive bibliography which has been previously published appeared in *L'Intermédiaire des Mathématiciens*, 1897, pp. 23–24, 86, 229; 1899, p. 179; 1901, p. 221; 1902, p. 82 and 1905, p. 112. The contributors were H. Bocard, A. R. Ericsson, E. B. Escott, A. Goulard, Langel and E. Lemoine. Another bibliography, by Max Simon, is given on p. 79 of his Ueber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert, Leipzig, 1906. The astounding inaccuracies throughout the book are here in evidence; for instance: when *Berton* is printed the reader is supposed to know that *Breton* (*de Champ*) was intended, also that *Eninger* stands for *Erchinger*.

zehneck im Kreise," *Archiv. f. Math. u. Phys.* (Grunert), vol. 42 (1864), pp. 361–374—R. Güntsche, "Geometrographische Siebzehnteilung des Kreises," *Archiv f. Math. u. Phys.* (3), vol. 4 (1903)—K. Hagge, "Einfache Behandlung der Siebzehnteilung des Kreises," *Zs. Math. Unterr.*, vol. 41 (1910), pp. 320–325—J. Hoüel, "Sur le polygone régulier de 17 côtés," [Exposition of von Staudt's construction], *Nouv. Annales de Math.*, vol. 16 (1857), pp. 310–311—S. Katayama, "The construction of a regular 17-sided polygon," *The Tôhoku Math. Journal*, vol. 4 (Feb., 1914), pp. 197–202—A. Padoa, "Poligoni regolari di 34 lati. Trattazione elementare," *Boll. di Mat.*, Bologna, vol. 2 (1903), pp. 2–10—W. Schoenborn, Elementare Beweise für einige Gleichungen, die Statt haben zwischen dem Radius eines Kreises, der Seite und der Diagonale der eingeschriebenen regulären 10-, 14-, 18-, 26-, 34-ecke, Pr. Krotoschin, 1873—Steggall, "The value of cos $2\pi/17$ expressed in quadratic radicals," *Proc. Edinb. Math. Soc.*, vol. 7 (1888–89), pp. 4–5.

Of additional references to general articles or books the following may be mentioned: B. Amiot, "Mémoire sur les polygones réguliers," *Nouv. Annales de Math.*, vol. 4 (1844), pp. 264–278—J. W. Butters, "On the solution of the equation $x^p - 1 = 0$ (p being a prime number)," *Proc. Edinb. Math. Soc.*, vol. 7 (1888–89), pp. 10–22—H. S. Carslaw, "Gauss's theorem on the regular polygons which can be constructed by Euclid's methods," *Proc. Edinb. Math. Soc.*, vol. 28 (1910), pp. 121–128—L. E. Dickson, "Constructions with ruler and compasses," in Monographs on Modern Mathematics (1911), 17-side: pp. 371–373—F. Giudice, "Sulla divisione del circolo," *Periodico di mat.* (3), vol. 9 (1912), pp. 161–169—F. Klein, "Elementarmathematik vom höheren Standpunkte aus," Teil I, Leipzig (1908), pp. 122 ff.—K. Kommerell, "Über die Konstruktion der regulären Polygone," *Math. Annalen*, vol. 72 (1912), pp. 588–592—I. L. A. Le Cointe, Leçons sur la théorie des fonctions circulaires et la trigonométrie, Paris (1858), p. 186f.—A. M. Legendre, Traité de trigonométrie at end of Eléments de géométrie, 8ème éd., Paris (1809), § VII, "Du polygone régulier de dix-sept côtés," pp. 419–421—J. Leslie, Elements of geometry, geometrical analysis, and plane trigonometry, second ed., Edinburgh (1811) p. 419 f.*

* It is also of interest to recall the passage in the letter which Sir William Rowan Hamilton wrote to De Morgan in 1852: "Are you *sure* that it is impossible to trisect the angle by Euclid? I have not to lament a single

In conclusion, let us consider approximate constructions of the regular 17-side.* In the seventeenth century C. Renaldini gave an interesting construction for any inscribed regular polygon.† It is as follows: "Construct on the diameter AB of a circumference C, an equilateral triangle ABD; divide AB into n equal parts; join the extremity E, of the second division, to the point D by the secant DEF, then AF is either exactly or approximately the length of the side of the required regular inscribed polygon of n sides." About two centuries later Housel considered the accuracy of this formula for values of n from 3 to 17;‡ for $n = 3$ or 4 or 6 the construction is exact; for $n = 17$ the angle subtended by AF at the center of the circle is about $36' \ 37''$ too large. Other approximations to the regular 17-side were given by Breton de Champ§ and Catalan|| and Postula.¶ Catalan pointed out that a closer

---

hour thrown away on the attempt, but fancy that it is rather a tact, a feeling, than a proof, which makes us think that the thing cannot be done. No doubt we are influenced by the cubic form of the algebraic equation. But would Gauss's inscription of the regular polygon of seventeen sides have seemed, a century ago, much less an impossible thing, by line and circle?"

De Morgan replied: "As to the trisection of the angle, Gauss's discovery increases my disbelief in its possibility. When $x^{17} - 1$ is separated into quadratic factors, we see how a construction by circles may tell. But, it being granted $ax^3 + bx^2 + cx + d$ is *not* separable into a real quadratic and a linear factor, I cannot imagine how a set of intersections of circles can possibly give no more or less than three distinct points." Graves' Life of Sir Wm. R. Hamilton, vol. 3 (1889), pp. 433–434.

The first rigorous proof of the impossibility of the problem of the trisection of an angle, under euclidean conditions, seems to have been given by L. Wantzel in 1837 (*Liouville*, tome 2, p. 369 f.).

* These are not discussed by Goldenring.

† De resolutione et compositione mathematica, Patavii, 1668, pp. 367–368. Renaldini considered that his construction was accurate in all cases. His error was first shown by Schultz in his Dissertatio de circuli divisione, Königsberg, 1691. (See S. Günther, *Österr. Zeitschrift für Realschulwesen*, vol. 3, pp. 523, 764.) For further history of this problem see A. G. Kästner, Geometrische Abhandlungen, Erste Sammlung, Göttingen, 1790, pp. 266–281; also *Zeits. f. Math. u. Naturw. Unterricht*, vol. 28 (1897), pp. 239, 252–255. Renaldini's approximate construction is sometimes attributed to Bion since it occurs in his Traité de la construction et des principaux usages des instruments de mathématiques, 4e éd., Paris, 1752.

‡ "Division pratique de la circonférence en parties égales," *Nouv. Annales de Math.*, vol. 12 (1853), pp. 77–79. See also remarks on this article by Tempier, *Nouv. Annales de Math.*, vol. 12 (1853), pp. 345–347; vol. 13 (1854), p. 295.

§ *Nouv. Annales de Math.*, vol. 5 (1846), pp. 226–227, 340.

|| Théorèmes et problèmes de géométrie élémentaire par H. C. de la Frémoire. Second édition entièrement revue et corrigée par E. Catalan, Paris, 1852, pp. 211–212.

¶ E. Catalan, Théorèmes et problèmes de géométrie élémentaire, 6e éd., Paris, 1879, p. 283.

approximation than that of Renaldini is found by taking for the side of the regular inscribed 17-side, one half the difference of the length of a side of the inscribed equilateral triangle and of a side of the regular inscribed hexagon. For the unit circle this leads to the length of a side of the 17-side as 0.36602, which differs from the correct value by about 0.001. According to Renaldini's construction the length is $1/\sqrt{7} = 0.37796\ldots$, which is about 0.02 too large.

R. C. ARCHIBALD.

BROWN UNIVERSITY,
PROVIDENCE, R. I.

SHORTER NOTICES.

Plane Trigonometry and Tables. Edited by GEORGE WENTWORTH and DAVID EUGENE SMITH. Ginn and Company, 1915. v + 188 + v + 104 pp. Price, $1.10.

THE formulation of the subject matter in the mind of the teacher largely determines the text he wants to use. This formulation is naturally the product of his experiences with texts studied and taught, and of his own cogitations on the subject and how he can most forcefully and successfully present it. The text under review fits into the plan of the reviewer for present purposes more happily than any of the many texts he has examined.

The authors state that as to sequence of material they have followed the rule "that the practical use of every new feature should be clearly set forth before the abstract theory is developed."

The six functions of acute angles are defined as ratios and put to practical use. The functions of complementary angles, and of angles of 30°, 45°, 60° are developed on pages 7 and 8 and put to immediate use. On page 23 appear the line definitions of the functions. The changes in the functions as the angle changes from 0° to 90° are exhibited through a figure giving the lines representing the six functions. Thus the student is brought to visualize the subject.

The natural trigonometric functions are employed just long enough to be known, and to cause the first twinges of the vexation of multiplication, when the subject of logarithms is

clearly set forth. Thereafter, the sine, cosine, tangent, and cotangent serve all purposes in computation. The more serious work with right triangles follows, extending to page 77.

The functions of *any* angle are introduced as ratios involving the coordinates of a point on the rotating line or terminal side of the angle; yet, as before, these functions are at once visualized as lines.

The advanced theoretical portions are well presented and ample for any student in his college course through the calculus. A clear open-faced index ends the text, pages 187–188.

The ten tables in the latter half of the book are so varied and so superior in arrangement and appearance as to be worth the full price of the book, as the list of tables that the student might regularly use in the future, even if he goes into any one of many lines of technical work.

This new revision has all the good features of the former texts and a spring-like, refreshing breath from the Napier Tercentenary in 1914 brought to it by Professor Smith. It is also a text that will be likely to "dominate the teaching of the subject" in the next and more exacting generation.

C. C. GROVE.

Modern Instruments and Methods of Calculation. A handbook of the Napier Tercentenary Exhibition. Edited by E. H. HORSBURGH with the cooperation of a committee. London, G. Bell & Sons, 1914. New York, The Macmillan Company. viii + 343 pp. 8vo. $1.90.

IT is very refreshing to read a book that provides so much more than its title leads one to expect, as does the volume under review. Even in its distinctively handbook features it gives copious notes concerning the special characteristics and history of the exhibits, the inventor, his period, etc. The editor states that "an endeavor has been made to make the Exhibition and Handbook useful to the laboratory computer, the engineer, the astronomer, the statistician, and to all who are interested in calculation," and success has crowned the effort in respect to the handbook at least.

There are a dozen sections lettered A to M, omitting J. In section A is an historical essay, reprinted from the *Proceedings* by permission of the Royal Philosophical Society of Glasgow, by Professor George A. Gibson on Napier's Life and Works.

Section C alone, on Mathematical Tables, would give the book a place in every mathematical reference library. It contains an account by Professor Cargill G. Knott, D.Sc., of Dr. Edward Lang's Logarithmic, Trigonometrical, and Astronomical Tables: A Working List of Mathematical Tables most conveniently arranged and amplified by notes by Herbert Bell, M.A. and J. R. Milne, D.Sc., both members of the editorial committee, as was also Professor C. G. Knott, its honorary secretary. There is also an historical essay by W. G. Smith on "Special development of calculating ability" that considers the psychology of calculating ability, with numerous references to the literature of the subject. Section D on Calculating Machines was written, or edited in part, by F. J. W. Whipple who presents the subject along the lines of the Catalogue raisonnée which he prepared for the Exhibition in connection with the Fifth International Congress of Mathematicians held at Cambridge, in 1912. His point of view is that "of the user of a machine who wishes to have a general idea of how it works rather than that of the expert who has to master every detail." He lets many of the manufacturers describe, through a scholarly expert, their own machines, hence I said, "edited in part." He outlines the prominent and essential mechanical means of performing the various arithmetical operations, and as an expert that he is, adds a paragraph on "The scope for improvement of calculating machines" that to the reviewer was of interest and inspiration.

It must be remembered that this is a catalog of the calculating machines on exhibition, not of all that have existed or are now on the market. With this in mind, attention was attracted by the sentences (pages 74, 75):

"In the first place, it is remarkable that no machine which does long multiplication automatically is on the market at present. . . . I fancy that it would not be difficult to modify the Thomas machine to enable it to act in this way."

On page 89 there follows, in another paper, the sentence:

"Two separate multiplications can be carried out at the same time by turning the handle."

These statements refer to quite different operations but they brought to the mind of the reviewer a vague recollection of having heard that the former process had been accomplished, possibly in some such way as described on pages 124–

125. They reminded him of experience with a Thomas machine in student days and of his wish then for a key-board instead of levers and for a carrying device for the quotient register. It may interest some readers to know that an American machine, the Monroe, supplies much of this that was left to be desired.

An article by P. E. Ludgate on "Automatic calculating machines," which will interest anyone who has seen a loom weaving the pictures of presidential candidates, say, into silk ribbon; an article by T. C. Hudson on "H. M. Nautical Almanac Office anti-difference machine"; and one on "Mathematical and calculating typewriters," complete section D.

Section E gives an abridgment of a classic article on "The abacus" by Dr. Cargill G. Knott, professor of physics, Imperial University of Tokyo, published in 1886 in the *Transactions of the Asiatic Society of Japan* and so almost unobtainable by us.

Space does not permit me further to enumerate the essays and notes, each written by a specialist and authoritative, concerning slide rules (25 pages), integraphs, integrometers, planimeters, their use in naval architecture, differentiating machine, harmonic analysers, tide-predicting machine, etc., but I cannot pass unnoticed Section H on Ruled Papers and Nomograms by the editor and Professor M. d'Ocagne respectively, which are parts of the literature to be read by anyone who would know the subject; also Section I (26 pages) on Mathematical Models, and Section K, a catalogue of portraits, engravings and medals of the collection of Prof. W. W. Rouse Ball, author of the well-known Short Account of the History of Mathematics.

The volume closes with a list of contributors and exhibitors, Section M. The names of the former are sufficient guarantee of the value of the articles. Their authority will secure for the book wide use amongst those for whom it was published.

C. C. GROVE.

Tables and Formulas (revised edition). By WILLIAM RAYMOND LONGLEY. Ginn and Company, 1915. 37 pp. Price, 50 cents.

THE author states in his preface: "This collection of tables and formulas is intended for use as a handbook for solving numerical problems in connection with the courses in mathematics in technical schools and colleges." It is a good collec-

tion for such a purpose. It will fill a long-felt need of both teacher and student, not only because the most frequently used tables and formulas, which are usually found scattered throughout half a dozen text-books, are here found within the covers of a booklet easily carried in one's pocket, but also because the more extended tables are usually too cumbersome, and give results to a much higher degree of accuracy than is commensurate with the data of the problem.

The book contains four place tables of common logarithms, and the natural and logarithmic values of trigonometric functions (interval of 1°), three-place tables of radian equivalents of degree measure and the natural values of trigonometric functions (interval of 0.05 radians). There are tables of squares and cubes from 1 to 100, square roots and cube roots from 1 to 1,000, reciprocals from 1 to 10 (interval of 0.1), Napierian logarithms from 1 to 10 (interval of 0.1) and from 10 to 100 (unit interval), and values of the exponential and hyperbolic functions from 0 to 10 (interval of 0.1). The collection of formulas contains the ordinary ones from algebra, geometry, trigonometry, analytic geometry, and the calculus, the last including some standard series, formulas for differentiation, and a well-chosen, well-arranged table of 177 integrals. The booklet concludes with the formulas for the solutions of the differential equations of harmonic motion, and damped and forced vibrations.

The reviewer would like to see the following formulas included in the collection: the length of the arc of a circle, $s = r\alpha$ (radians), the products like $2 \sin u \cos v = \sin (u + v) + \sin (u - v)$, and the area of a triangle in terms of the three sides. One serious defect of the table of trigonometric integrals is that these are given as $\int \sin x dx$, etc., instead of $\int \sin ax dx$, etc., especially since $\int \sin u du$, etc., are nowhere given; the forms $\int \tan^n ax \sec^2 ax\, dx$, etc., should also be included since the form $\int u^n du$ is not given. Mention should also be made of the method of integrating rational fractions.

The type is clear and the tables are easily read. The many excellent qualities of the booklet, together with its small price, will commend it to the students in the mathematics courses in colleges and technical schools.

JOSEPH LIPKA.

Vorlesungen über projektive Geometrie. By FEDERIGO ENRIQUES. Second German edition by H. FLEISCHER. Leipzig, Teubner, 1915. xiv + 354 pp. Price (cloth), 10 Marks.

THE first German translation (with a prefatory note by Klein) of Enriques's lectures on projective geometry appeared in 1903 and was ably reviewed in the BULLETIN* by Professor Virgil Snyder. As the second edition does not contain any essential changes, not much needs to be added to that review. In view of the considerable advances that have been made in this field during the past two decades, a fuller discussion of the various aspects of the fundamental theorem of projective geometry would have been of great assistance to the student. At any rate the otherwise well written historic-critical note at the end of the treatise might have been brought up to date in this respect. In the discussion of projective coordinates, pages 332–336, which is essential for a proper understanding of the one-to-one correspondence between an analytic space (x_1, x_2, x_3, x_4) and the space based upon and abstracted from intuition (parabolic, hyperbolic, or elliptic), in which the points are defined by means of cross-ratios, a detailed proof of this correspondence would be commensurate with its importance.

From a didactic standpoint Enriques's "lectures" can still be recommended as an excellent introduction to the subject.

ARNOLD EMCH.

Descriptive Geometry. By H. W. MILLER, head of the department of general engineering drawing in the University of Illinois. New York, Wiley and Son, third edition, 1915. 149 pages, 86 figures and 8 quiz sheets.

Descriptive Geometry for Students in Engineering Science and Architecture. A carefully graded course of instruction, by HENRY C. ARMSTRONG, associate professor of descriptive geometry and drawing, McGill University. New York, Wiley and Son, 1915. vi + 125 pages and 114 figures.

Darstellende Geometrie, von MARCEL GROSSMANN, professor at the technical school of Zurich. Leipzig, Teubner, 1915. v + 137 pages and 109 figures.

THE first sentence of the preface to Professor Miller's book reads: "Believing that no one study plays a larger part than

* Vol. 10, pp. 355–58, April, 1904.

descriptive geometry in the shaping of the student's mind into the analytic thinking machine, necessary to success in any engineering profession, the author has outlined and written the text with this as its chief aim."

Minute instructions as to lettering, trimming, weighting lines, notation, methods of study precede the subject proper. The variety of type and prominence of figures leave no question of clearness, but the page has in places a striking resemblance to a bill board in consequence. The method of representing a point, line, plane are explained n great detail. It provokes a smile to read on page 16, "Axiom: The two projections of a point must be on the same perpendicular to GL" and similar incidents on page 24, in which one statement is followed by a proof. No exercises are given for the student, and no numerical cases are worked out at all. This defect is partly remedied by a series of eight quiz sheets with drawings, put at the end of the book.

The chapter on revolution, general profile, and problems relating to them is longer than the others. The figure representing the perspective of a circle is hardly necessary at this stage and is rather too hard to understand. Most of the theorems are well explained, but their inter-relations are not well brought out. The chapter (Chapter 6) on lines and surfaces would hardly bear mathematical analysis. The classification of lines and surfaces is a curiously arbitrary one, which would greatly confuse a bright student. Single and double curvature contact are not defined, and are crudely employed. The definitions of a ruled surface, developable, and double curved surface are such as to apply only to the few elementary illustrations employed, yet this fact is nowhere stated. The concept of a tangent plane to the ellipsoid should at least have been shown to be unique. The intersections of certain surfaces with planes is more satisfactorily discussed, as is also the development of cylinders and cones. Thus far 95 pages have been covered, which includes all of descriptive geometry that is treated. Chapter 10 is on shades and shadows—well written from an architectural or structural standpoint—it contains no new mathematics. The last chapter is on perspective. It brings in no new mathematical principles; it treats of a number of elementary properties in a rather empirical way, totally overlooking the beautiful transition from descriptive to projective geometry.

The reviewer is not competent to speak of the merits of the book from the engineer's viewpoint, but had the text been more mathematical and a more scientific aim kept in mind, the claims of the first sentence of the preface would have been much more generally fulfilled.

Professor Armstrong's book shows many contrasts with the preceding. It contains no mechanical instructions for the reader; it does not use bold-faced type, and very little explanation is given concerning different kinds of lines. The book is full of exercises, an appropriate list being furnished at intervals of a few pages. In fact, so little explanation is given that a student without a competent teacher would have to exercise considerable patience to master the text. All the steps are given, but in a very concise form. Thus, restricted positions of lines, planes, plane figures, polyhedra, involving shadows on the horizontal and vertical planes, are all treated in forty pages.

Part II, which treats of planes, lines, and points in unrestricted position, is less concise. The discussion is clear, is usually mathematically correct, and the frequent exercises allow the reader to test his grasp of every point. What seems an objection is that too many figures are prepared, the drawings given are completed, before the student can understand how it was done. Of course this difficulty is at once obviated by a good teacher. It is unusual to meet with axometric projection as early in the development (page 69) as in the present book. It is also presented in an unusual way, namely, as the ordinary horizontal and vertical projections of the corner and the edges of a cube, the idea of scale being developed later. The explanations are clear, but the whole treatment of this section (8 pages) is too brief to be of much use. But the last chapter of Part II, devoted to sections of simple solids and traces of cylinders and cones is excellently well done. The figures alone completely show the whole process, and they are supplemented by a brief description and followed as usual by exercises for the student.

Part III begins with problems involving tangent planes to cones and cylinders and their applications to sections of solids; tangent planes of a sphere, common to two or three spheres. The text is almost always correct mathematically—only such problems are considered in which the tangent line to a conic

can be constructed geometrically. Then follow simple intersections and their developments; here the treatment is the usual one. One modest case of a screw-thread is worked out. Finally, a short chapter on perspective is given; it is very clear as far as it goes, and includes some excellent examples for the student, but it stops just as the student's interest is aroused. Perhaps this is first class pedagogy, but somewhat dangerous without a good teacher.

Although the title of Professor Grossmann's text could hardly convey less information than it does, this book s not for beginners—indeed it presupposes a fairly comprehensive course in orthographic descriptive geometry and considerable familiarity with the technique of mechanical drawing. Its purpose is more specifically mathematical, to explain the meaning of the processes employed, and to compare their merits. The first discussion is to show that the ground line is not needed, and that any line in a plane of projection may be used as new ground line. Then follows a carefully written discussion of axonometry. If it could have been supplemented with a generous list of appropriate exercises, what a fine presentation it would make! It includes a good demonstration of Polke's theorem, that any three segments on three arbitrary concurrent lines can be taken as the projections of three concurrent edges of a cube. The chapter on perspective is less satisfactory; the details are clear enough but the purpose of it all is not as clearly presented as it might be, metrical details coming in unusually early in the discussion. Later the problem develops in a more interesting way. An unusual theme is a full discussion of stereographic projection. A dozen pages are devoted to the interesting topic of photogrammetry. Both inner and outer position are treated, and reconstruction from two vertical photographs, or from two oblique ones.

The second part, curves and surfaces, begins with a considerable digression on the analytic theory of plane and space curves, including parametric representation and differential properties. As is too liable to be the case, this attempt has but little purpose. It is too brief really to teach one unfamiliar with the ideas concerned, and unnecessary to one already acquainted with it. In the corresponding treatment of curved surfaces, a number of theorems are stated without

references or attempts at proofs. After a very brief discussion of the simple helix, the first surfaces discussed are a topographical map and forms of embankments and excavations. Granted this is important to one learning uses of graphical methods, a less felicitous application of the earlier theory could hardly be devised. Cones and cylinders fare rather better, as they connect directly with the theory. Plane perspective is developed from the standpoint of geometric correspondence; use is made of cross-ratio, and a fairly full discussion of conics from the Steiner construction is given, including the theorems of Pascal and Brianchon, and a few applications.

The treatment of intersections of cones and cylinders is rather brief; space quartics (of the first kind) and space cubics are considered and a few examples given. Plane sections of surfaces of revolution, and illumination are next discussed. From the three-page description the average reader can expect but a very vague and indefinite idea of a ruled surface. In one line the half-dual property is disposed of. Nearly five pages are given to the helicoid, six to the ruled quadrics, and three to non-ruled quadrics. At the end of the volume is a list of a dozen other texts for references; all of them have been reviewed in the BULLETIN.

While it would certainly be desirable to have students of geometry in the technical schools and colleges familiar with the topics here cited, I cannot believe that the best way to accomplish that purpose is to attempt to acquire the necessary knowledge in such a condensed way.

VIRGIL SNYDER.

NOTES.

THE opening (January) number of volume 17 of the *Transactions of the American Mathematical Society* contains the following papers: "On functions of several complex variables," by W. F. OSGOOD; "A study of certain functional equations for the ϑ-functions," by E. B. VAN VLECK and F. H'DOUBLER; "A set of four independent postulates for Boolean algebras," by B. A. BERNSTEIN; "Transformations of surfaces Ω (second memoir)," by L. P. EISENHART; "On figures of equilibrium of a rotating compressible fluid mass; certain negative results," by E. J. MOULTON.

THE concluding (December) number of volume 1 of the *Proceedings of the National Academy of Sciences* contains the following mathematical papers: "Theorem concerning the singular points of ordinary linear differential equations," by G. D. BIRKHOFF; "Definition of limit in general integral analysis," by E. H. MOORE. The volume contains in all 21 articles on mathematics.

THE following papers on mathematics or mathematical physics have recently appeared in the *Proceedings of the American Academy of Arts and Sciences:* "Geometry whose element of arc is a linear differential form, with application to the study of minimum developables," by C. L. E. MOORE, volume 50, pages 197–222; "Expansion problems with irregular boundary conditions," by DUNHAM JACKSON, volume 51, pages 381–417; "The mechanics of telephone-receiver diaphragms as derived from their motional-impedance circles," by A. E. KENNELLY and H. A. AFFEL, volume 51, pages 419–482.

ON December 30 and 31, 1915, there was held at Columbus, Ohio, the organization meeting of a new mathematical association, the call for which had been signed by 450 persons representing every state in the Union, the District of Columbia, and Canada. The object of the new Association is to assist in promoting the interests of mathematics in America, especially in the collegiate field. It is not intended to be a rival of any existing organization, but rather to supplement the Secondary Associations on the one hand, and the American Mathematical Society on the other; the former being well organized and effective in their field, and the latter having definitely limited itself to the field of scientific research. In the field of collegiate mathematics, however, there has been, up to this time, no organization and no medium of communication among the teachers, except the *American Mathematical Monthly,* which for the past three years has been devoted to this cause. The new organization, which has been named the Mathematical Association of America, has taken over the *American Mathematical Monthly* as its official journal.

There were 104 persons present at the organization meeting. The constitution and by-laws together with a full report of the proceedings have been published in the January issue of the *Monthly*. The following officers were elected: President, E. R.

Hedrick. First Vice-President, E. V. Huntington; Second Vice-President, G. A. Miller; Secretary-Treasurer, W. D. Cairns; Publication Committee, H. E. Slaught, W. H. Bussey, and R. D. Carmichael.

These officers, together with the following, constitute the Executive Council: R. C. Archibald, Florian Cajori, B. F. Finkel, D. N. Lehmer, E. H. Moore, R. E. Moritz, M. B. Porter, K. D. Swartzel, J. N. Van der Vries, Oswald Veblen, J. W. Young, Alexander Ziwet.

AT the annual meeting of the London mathematical society held November 11, the following papers were read: By G. H. HARDY, "The second theorem of consistency for summable series; Weierstrass's non-differentiable series"; by F. B. PIDDICK, "The kinetic theory of the motion of ions in gases"; by H. W. TURNBULL, "Some singularities of surfaces and their differential geometry"; by J. W. CAMPBELL, "Periodic solutions of the problem of three bodies in three dimensions"; by C. R. DINES, "Functions of positive type and related topics in general analysis"; by C. H. YEATON, "Surfaces characterized by special properties of their directrix congruences." At the meeting of December 9 the following papers were read: By H. JEFFREYS, "The vibrations of a special type of dissipative system"; by E. J. W. WHIPPLE, "Diffraction by a wedge"; by T. L. WREN, "Some applications of the two-three birational space transformation"; by T. C. LEWIS, "The circles which touch the escribed circles of a triangle."

AT the meeting of the Edinburgh mathematical society on December 10 the following papers were read: "Real linear substitutions with equimodular multipliers," by D. G. TAYLOR; "On the linear differential equation of the second order," by S. BRODETSKY; "Fourier's integral," by T. A. BROWN.

THE annual meeting of the British mathematical association was held at the London day training college on January 5. The following papers were read at this meeting: By A. N. WHITEHEAD, "The aims of education, a plea for reform" and "The allowance for the earth's rotation in the theory of projectiles"; by G. W. PALMER, "The results of an investigation into the degree of accuracy that may be expected in simple arithmetical work in boys' schools"; by A. LODGE, "Discussion on the use of mathematical tables; desiderata of such tables."

THE royal society of Bologna announces the following prize problems for 1916:

"Set forth by critical and historical methods, the organic development of the theory of elliptic functions, including the different points of view under which the theory has been considered from the end of the eighteenth century to the present time. Indicate the influences which these various points of view have had on other branches of analysis."

"From the beginning of the twentieth century it has been proposed to substitute new definitions for the classic definition of a definite integral, with the purpose of generalizing the notion of an integral and of applying it to classes of functions as extended as possible. It is proposed to submit these various definitions to a critical and historical analysis, and to recognize those definitions which one would preferably adopt together with an exhaustive justification for the choice made."

Competing memoirs should be submitted to the secretary of the society under the usual conditions before December 31, 1916. The value of the prize is 500 lire.

THE royal medal of the Royal society of London has been awarded to Professor Sir J. LARMOR for investigations in mathematics and physics.

THE firm of Macmillan in New York announce that a book on the theory of errors and least squares, by L. D. WELD, of Coe College, Iowa, is in the press, and will be issued in February.

AT Harvard University Dr. DUNHAM JACKSON has been promoted to an assistant professorship of mathematics.

PROFESSOR P. VOGEL, of the war academy of Munich, died in October, at the age of 58 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ARCHIBALD (R. C.). Euclid's Book on Divisions of Figures (*περὶ διαιρέσεων βιβλίον*) with a restoration based on Woepcke's text and on the Practica Geometriae of Leonardo Pisano. Cambridge, University Press, 1915. 8+88 pp. 6s.

BAGNERA (G.). Lezioni di calcolo infinitesimale. Disp. 36–47 (ultima). Palermo, tip. Matematica, 1915. 8vo. Pp. 12+281–375.

BOLZANO (B.). Wissenschaftslehre. Neu herausgegeben von A. Höfler. 2ter Band. Leipzig, F. Meiner, 1915. M. 12.00

CARMICHAEL (R. D.). Diophantine analysis. (Mathematical monographs, No. 16.) New York, Wiley, 1915. 8vo. 6+118 pp. Cloth. $1.25

CIAMPOLINI (S.). Sulla teoria delle curve nello spazio ellitico. Pisa, tip. succ. fratelli Nistri, 1915. 8vo. 52 pp.

EUCLID. See ARCHIBALD (R. C.).

HAFNER (H.). Deformation einer geradlinigen Fläche unter der Bedingung, dass die Erzeugenden Erzeugende bleiben und die Längen einer aequidistanten Kurvenschar unverändert erhalten werden. (Diss., Techn. Hochschule, München.) München, G. Hafner, 1914.

HÖFLER (A.). See BOLZANO (B.).

JACOBSTHAL (W.). See WEBER (H.).

KNOBLAUCH (J.). See WEIERSTRASS (K.).

KRAFFT (M.). Zur Theorie der Faberschen Polynome und ihrer zugeordneten Funktionen. (Diss.) Marburg, 1915.

KRAZER (A.). Zur Geschichte der graphischen Darstellung von Funktionen. Karlsruhe, J. Langs, 1915.

LEIB (D. D.). Problems in the calculus with formulas and suggestions. Boston, Ginn, 1915. 12+224 pp. Cloth. $1.00

LEONARDO PISANO. See ARCHIBALD (R. C.).

PHILLIPS (H. B.). Analytic geometry. New York, Wiley, 1915. 8vo. 7+197 pp. Cloth. $1.50

RUNGE (C.). Mathematik und Bildung. Festrede im Namen der Georg-August-Universität am 9. Juni 1915 gehalten. Göttingen, Dieterich, 1915.

S. J. Drei Gleichungen als Grundlage für einen Beweis des sogenannten grossen Satzes von Fermat allgemein-verständlich vorgeführt. Darmstadt, K. Köhler, 1915. 16 pp. M. 1.20

WEBER (H.) und WELLSTEIN (J.). Enzyclopädie der Elementar-Mathematik. Ein Handbuch für Lehrer und Studierende. 2ter Band: Elemente der Geometrie. Bearbeitet von H. Weber, J. Wellstein und W. Jacobsthal. 3te Auflage. Leipzig, Teubner, 1915. M. 12.00

WEIERSTRASS (K.). Mathematische Werke. 5ter Band: Vorlesungen über die Theorie der elliptischen Funktionen. Bearbeitet von J. Knoblauch. Berlin, Mayer und Müller, 1915.

WELLSTEIN (J.). See WEBER (H.).

WIARDA (G.). Ueber gewisse Integralgleichungen erster Art, besonders aus dem Gebiete der Potentialtheorie. (Diss.) Marburg, 1915.

WILDBRETT (A.). Algebraische Analysis, Algebra und Infinitesimalrechnung. Lehrbuch mit Aufgabensammlung für die Oberstufe von Realanstalten. 2ter Teil: Infinitesimalrechnung. Nürnberg, Korn, 1915. 199 pp. M. 3.80

——. Analytische und projektive Geometrie. Lehrbuch mit Aufgaben-

sammlung für die Oberstufe von Realanstalten. 1ter Teil: Analytische Geometrie der Geraden und des Kreises. Elemente der projektiven Geometrie. 2ter Teil: Analytische und projektive Geometrie der Kegelschnitte. Nürnberg, Korn, 1915. 144+143 pp M. 2.90+2.90

WOEPCKE (F.). See ARCHIBALD (R. C.).

WRIGHT (W. C.). Wright vs. Eratosthenes or observations on the law or conditions of sequence of prime numbers, with a statement of a new and short method of determining them, and illustrative tables and figures. Boston, Wright, 1915. 8vo. 15 pp. $0.25

II. ELEMENTARY MATHEMATICS.

BÖRNER (E.). See SUPPANTSCHITSCH (R.).

CAJORI (F.) and ODELL (L. R.). Elementary algebra. First year course. New York, Macmillan, 1915. 8vo. 8+206 pp. Cloth.

CHAPPELL (E.). Five figure mathematical tables. Edinburgh, Chambers, 1915. 16+320 pp. 5s.

DRAEGER (M.). See LIETZMANN (W.).

DÜSING (K.). See SCHULTZ (E.).

FAZZARI (G.). Elementi di aritmetica con note storiche e numerose questioni varie per le scuole medie superiori. Parte prima. 3a edizione. Palermo, Trimarchi, 1916. 8vo. 6+134 pp. L. 1.60

GLASER (R.). Stereometrie. 3te verbesserte Auflage. (Sammlung Göschen.) Berlin, Göschen, 1915. M. 0.90

HIRTS Kriegs-Rechenbuch. Stoff- und Aufgabensammlung zum Weltkrieg 1914–1915. Leipzig, Hirt, 1915. 40 pp. M. 0.50

HOLMAN (S. W.). Computation rules and logarithms. 5th reprint. New York, Macmillan, 1913. 77 pp. $1.00

JACKSON (C. S.). A twentieth century arithmetic. London, Dent, 1915. 8vo. 3s. 6d.

KANDEL (I. L.). The training of elementary school teachers in mathematics in the countries represented in the International commission on the teaching of mathematics. (United States Bureau of Education Bulletin, 1915, No. 39.) Washington, Government Printing Office, 1915. 56 pp.

KRÜGER (C.). Laerbog i Mathematik. 1. Band: Arithmetik og Algebra for Gymnasiets mathematisk-naturvidenskabelige Linie. Kopenhagen, G. E. C. Gads Ferlag, 1914. 83 pp. Kr. 1.75

LICHTBLAU (W.) und WIESE (B.). Mathematisches Unterrichtswerk für Lehrerbildungsanstalten. Neubearbeitung. 1te Abteilung: Rechnen, Arithmetik und Algebra. 3te Auflage. Breslau, Hirt, 1915. 352 pp. M. 4.00

LIETZMANN (W.). Die Ausbildung der Mathematiklehrer an den höheren Schulen Deutschlands. (Berichte und Mitteilungen, veranlasst durch die Internationale Mathematische Unterrichtskommission. Erste Folge, XI.) Leipzig, Teubner, 1915. Pp. 311–328. M. 0.60

LIETZMANN (W.) und DRAEGER (M.). Ergebnisse zu Bardey-Lietzmann, Aufgabensammlung für Arithmetik, Algebra und Analysis. 1ter Teil: Unterstufe. Leipzig, Teubner, 1915. 63 pp.

LÖWENHAUPT (V.). Der grosse Krieg in Zahlen. Eine Ergänzung zu den Rechenbüchern. Leipzig, Teubner, 1915. 48 pp. M. 0.80

NUNN (T. P.). Exercises in algebra (including trigonometry). Part 2, with answers. (Longman's Modern Mathematical Series.) 8vo. 12+551 pp. 6s. 6d.

ODELL (L. R.). See CAJORI (F.).

REIDT (F.). Sammlung von Aufgaben und Beispielen aus der Trigonometrie und Stereometrie. 2ter Teil: Stereometrie. 5te Auflage. Neu bearbeitet von H. Thieme. Leipzig, Teubner, 1914. M. 3.80

RUTGERS (W.). Beitrag zur Weiterentwicklung der Algebra. Als Manuskript herausgegeben durch die Erben des Verfassers. Oerliken bei Zürich, F. J. Rutgers, 1915.

SCHNEIDER (A.). Lehr- und Uebungsbuch der Geometrie. Für Lehrer- und Lehrerinnenbildungsanstalten. 2ter Teil: Ebene Trigonometrie, Stereometrie und sphärische Trigonometrie. Leipzig, Freytag, 1915. M. 3.40

SCHULTZ (E.) und DÜSING (K.). Leitfaden der Stereometrie für höhere technische Lehranstalten und zum Selbstunterricht, nebst einer Sammlung von Aufgaben aus den Gebieten der Praxis. Essen, 1915. M. 2.00

SMITH (D. E.). The teaching of elementary mathematics. 9th reprint. New York, Macmillan, 1914. 312 pp. $1.00

——. See WENTWORTH (G.).

SUPPANTSCHITSCH (R.). Mathematisches Unterrichtswerk; für Mädchenlyzeen bearbeitet von E. Börner. Geometrie. 5ter Teil: Stereometrie. Wien, Tempsky, 1915. 43 pp. Kr. 1.20

THIEME (H.). See REIDT (F.).

UMLAUF (K.). Der mathematische Unterricht an den Seminaren und Volksschulen der Hansestädte. (Abhandlungen über den mathematischen Unterricht in Deutschland, Band V, Heft 5.) Leipzig, Teubner, 1915. 7+160 pp. M. 4.80

WENTWORTH (G.) and SMITH (D. E.). Plane and spherical trigonometry and tables. Boston, Ginn, 1915. 6+230+26+6+104 pp. Cloth. $1.35

WIESE (B.). See LICHTBLAU (W.).

III. APPLIED MATHEMATICS.

ALBRECHT (T.). Ergebnisse der Breitenbeobachtungen auf dem Observatorium in Johannisburg von März 1910 bis März 1913. Berlin, 1915. M. 4.00

BARTON (E. H.). An introduction to the mechanics of fluids. New York, Longmans, 1915. 8vo. 14+249 pp. $1.75

BJERKNES (C. A.). Hydrodynamische Fernkräfte. 5 Abhandlungen über die Bewegung Kugelförmiger Körper in einer inkompressiblen Flüssigkeit (1863–1880). Herausgegeben von A. Korn und N. Bjerknes. Leipzig, 1915. M. 5.60

BJERKNES (N.). See BJERKNES (C. A.).

BOCCARDI (G.). Elementi di astronomia: supplemento alla parte I. Torino, tip. s. Giuseppe degli Artigianelli, 1915. 8vo. 404 pp. L. 3.50

CAMPBELL (W. W.). Elements of practical astronomy. 4th reprint. New York, Macmillan, 1913. 253 pp. $2.00

CASTELLO VIDAL (P.). La estrofoide y el problema del billar circular. Barcelona (*Mem. Acad.*), 1914. 4to. 18 pp. M. 2.00

FURNESS (C. E.). An introduction to the study of variable stars. Boston, Houghton-Mifflin, 1915. 20+327 pp. $1.75

HOSKINS (L. M.). Elements of graphic statics. 2d edition. New York, Macmillan, 1914. 200 pp. $2.25

HOWE (G.). Mathematics for practical men. New York, Van Nostrand, 1915. 143 pp. $1.25

KORN (A.). See BJERKNES (C. A.).

KRIEMLER (C.). Technische Mechanik. Ein Lehrbuch der Statik und Dynamik starrer und nachgiebiger Körper. Stuttgart, 1915. M. 14.00

KULLRICH (E.). Mathematisch-physikalische Tafeln. 2te Auflage. Leipzig, Teubner, 1915. 12 pp. M. 0.60

McKAY (R. E.). The theory of machines. London, Arnold, 1915. 15s.

MILHAM (W. I.). Meteorology. Reprint. New York, Macmillan, 1914. 549 pp. $4.50

MOULTON (F. R.). An introduction to astronomy. 7th reprint. New York, Macmillan, 1914. 557 pp. $1.60

NETTMANN (P.). Der Torsionsindikator. Berlin, 1915. M. 5.00

PANOFSKY (E.). Dürers Kunsttheorie vornehmlich in ihrem Verhältnis zur Kunsttheorie der Italiener. Berlin, Reimer, 1915. 209 pp.

PFLIEGER-HAERTEL (H.). Ueber die kleinen Schwingungen einer dreigliedrigen ebenen Gelenkkette, zugleich ein Beitrag zur Theorie der einfachen Hebelwage. (Diss.) Jena, 1914.

POPPLEWELL (W. C.). The elements of surveying and geodesy. New York, Longmans, 1915. 8vo. 12+244 pp. $2.25

RUTHS (C.). Neue Relationen im Sonnensysteme und Universum. Darmstadt, C. Ruths, 1915.

SKOPIK (O. L.). Wie berechnet, konstruiert und baut man ein Flugzeug? 2te unveränderte Auflage. Berlin, 1915. M. 6.00

STEPHENS (J. S.). Theory of measurements. London, Constable, 1915. 7+81 pp. 6s.

WATERBURY (L. A.). A vest pocket handbook of mathematics for engineers. New York, Wiley, 1915. 10+213 pp. $1.50

WILDBRETT (A.). Darstellende Geometrie. Lehrbuch mit Aufgabensammlung. Nürnberg, Korn, 1915. 1ter Teil: Schiefe Parallelprojektion und Orthogonalprojektion. 2ter Teil: Kegel, Zylinder und Kugel. Einführung in die Perspektive. 147+101 pp. M. 3.20+2.00

ZEHNDER (L.). Mathematische Zusätze zum Grundriss der Physik. 2te Auflage. Tübingen, H. Laupp, 1915. M. 0.40

THE TWENTY-SECOND ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-second annual meeting of the Society was held in New York City on Monday and Tuesday, December 27–28, 1915. The attendance at the four sessions included the following seventy-two members:

Mr. J. W. Alexander II, Professor Clara L. Bacon, Dr. Ida Barney, Professor R. D. Beetle, Mr. D. R. Belcher, Dr. A. A. Bennett, Professor G. D. Birkhoff, Professor E. W. Brown, Dr. T. H. Brown, Dr. R. W. Burgess, Professor F. N. Cole, Professor J. L. Coolidge, Professor D. R. Curtiss, Professor L. E. Dickson, Professor John Eiesland, Professor L. P. Eisenhart, Professor T. C. Esty, Professor F. C. Ferry, Dr. C. A. Fischer, Professor W. B. Fite, Professor Tomlinson Fort, Dr. Meyer Gaba, Mr. W. Van N. Garrettson, Dr. G. M. Green, Professor C. C. Grove, Professor J. G. Hardy, Professor H. E. Hawkes, Dr. Olive C. Hazlett, Professor E. R. Hedrick, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. Edward Kircher, Professor J. K. Lamond, Dr. D. D. Leib, Dr. P. H. Linehan, Dr. Joseph Lipka, Professor W. R. Longley, Professor C. R. MacInnes, Dr. W. E. Milne, Professor H. H. Mitchell, Professor C. L. E. Moore, Dr. R. L. Moore, Professor F. M. Morgan, Professor Frank Morley, Professor G. D. Olds, Professor W. F. Osgood, Dr. Alexander Pell, Dr. G. A. Pfeiffer, Professor H. B. Phillips, Professor Arthur Ranum, Professor L. H. Rice, Professor R. G. D. Richardson, Dr. P. R. Rider, Mr. J. F. Ritt, Professor J. E. Rowe, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor C. G. Simpson, Professor Mary E. Sinclair, Professor Clara E. Smith, Professor P. F. Smith, Professor Sarah E. Smith, Professor W. M. Smith, Professor Virgil Snyder, Professor Elijah Swift, Mr. H. S. Vandiver, Professor E. E. Whitford, Dr. C. E. Wilder, Professor Ruth G. Wood, Professor J. W. Young.

The President of the Society, Professor E. W. Brown, occupied the chair, being relieved by Professor Edward Kasner. The Council announced the election of the following persons to membership in the Society: Professor W. E. Edington, University of New Mexico; Professor J. L. Gibson, University

of Utah; Dr. W. E. Milne, Bowdoin College; Professor L. J. Reed, University of Maine. Nine applications for membership in the Society were received.

The total membership of the Society is now 732, including 73 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 418; the number of papers read was 197. The number of members attending at least one meeting during the year was 253. At the annual election 204 votes were cast. The Treasurer's report shows a balance of $10,470.58, including the life membership fund of $5,560.30. Sales of the Society's publications during the year amounted to $1,832.93. The Library now contains about 5,250 volumes, excluding unbound dissertations.

The Society received with great regret the resignation of Professor L. E. Dickson from the Editorial Committee of the *Transactions*, to take effect October 1, 1916, at the end of fifteen years of editorial services including six years as member of the Editorial Committee. Committees were appointed by the Council to nominate successors to Professor Dickson and Professor D. R. Curtiss, whose first term as member of the Editorial Committee expires on October 1, 1916.

Sixty members and friends attended the annual dinner of the Society on Monday evening.

At the annual election, which closed on Tuesday morning, the following officers and other members of the Council were chosen:

Vice-Presidents,	Professor E. R. Hedrick,
	Professor Virgil Snyder.
Secretary,	Professor F. N. Cole.
Treasurer,	Professor J. H. Tanner.
Librarian,	Professor D. E. Smith.

Committee of Publication,
Professor F. N. Cole,
Professor Virgil Snyder,
Professor J. W. Young.

Members of the Council to Serve until December, 1918,

Professor G. A. Bliss,	Professor W. B. Fite,
Professor R. D. Carmichael,	Professor F. S. Woods.

The following papers were read at this meeting:

(1) Mr. J. F. RITT: "On the derivatives of a function at a point."

(2) Mr. J. F. RITT: "The finite groups of a class of functions of a real variable."

(3) Professor J. E. ROWE: "A new method of deriving the equation of a rational plane curve from its parametric equations."

(4) Professor H. B. PHILLIPS: "Elastic nets."

(5) Professor ARTHUR RANUM: "The singular points of analytic space curves."

(6) Dr. H. M. SHEFFER: "The reduction of non-monadic relations to monadic" (preliminary communication).

(7) Dr. H. M. SHEFFER: "The elimination of modular existence postulates."

(8) Professor BESSIE I. MILLER: "A new canonical form of the elliptic integral."

(9) Mr. A. R. SCHWEITZER: "On the use of supernumerary indefinables in the construction of axioms."

(10) Professor TOMLINSON FORT: "Linear difference and differential equations."

(11) Dr. DUNHAM JACKSON: "Algebraic properties of self-adjoint systems."

(12) Professor G. D. BIRKHOFF: "On dynamical systems with two degrees of freedom."

(13) Professor G. D. BIRKHOFF: "Infinite products of analytic matrices."

(14) Professor E. B. WILSON: "Ricci's absolute calculus and its application to the theory of surfaces."

(15) Professor C. L. E. MOORE: "Some theorems regarding two-dimensional surfaces in euclidean n-space."

(16) Dr. OLIVE C. HAZLETT: "On the fundamental invariants of nilpotent algebras in a small number of units."

(17) Dr. EDWARD KIRCHER: "Some properties of finite algebras."

(18) Professor M. FRÉCHET: "On Pierpont's definition of integrals."

(19) Professor EDWARD KASNER: "Infinite groups of conformal transformations."

(20) Dr. JOSEPH LIPKA: "Isogonal, natural, and isothermal families of curves on a surface."

(21) Dr. L. L. SILVERMAN: "On the consistency and

equivalence of certain generalized definitions of the limit of a function of a continuous variable."

(22) Professor L. P. EISENHART: "Ruled surfaces generated by the motion of an invariable curve."

(23) Professor L. P. EISENHART: "Transformations of surfaces Ω (second paper)."

(24) Dr. G. M. GREEN: "On rectilinear congruences and nets of curves on a surface."

(25) Professor W. F. OSGOOD: "On infinite regions."

(26) Professor JOHN EIESLAND: "On sphere-flat geometry."

(27) Professor J. L. COOLIDGE: "The meaning of Plücker's numbers for a real curve."

(28) Professor W. M. SMITH: "Characterization of the trajectories described by a particle moving under central force varying inversely as the nth power of its distance from the center of force."

(29) Professor H. H. MITCHELL: "On the generalized Jacobi-Kummer cyclotomic function."

(30) Professor H. H. MITCHELL: "On the congruence $cx^\lambda + 1 \equiv dy^\lambda$ in a Galois field."

(31) Professor R. D. BEETLE: "Sets of properties characteristic of the arithmetic and geometric means."

(32) Dr. R. L. MOORE: "On the foundations of geometry."

(33) Dr. W. C. GRAUSTEIN: "The correspondence of space curves by the transformation of Combescure and by a transformation thereby suggested."

(34) Mr. R. E. GLEASON: "On Dirichlet's principle."

(35) Dr. W. E. MILNE: "On the degree of convergence of Birkhoff's series."

(36) Professor G. C. EVANS: "A generalization of Bôcher's analysis of harmonic functions" (preliminary report).

(37) J. W. ALEXANDER, II: "On the factorization of plane Cremona transformations."

(38) Mr. L. B. ROBINSON: "On elimination between several polynomials in several variables."

Professor Fréchet's paper was communicated to the Society through Professor Curtiss; Mr. Gleason was introduced by Professor Birkhoff. In the absence of the authors Professor Wilson's paper was read by Professor C. L. E. Moore, and the papers of Dr. Sheffer, Professor Miller, Mr. Schweitzer, Professor Fréchet, Dr. Silverman, Professor Coolidge, Dr. Graustein, Professor Evans, Mr. Alexander, and Mr. Robinson were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Mr. Ritt showed in an earlier paper that, if x is confined to the real domain, the function

$$\sum_{n=1}^{\infty} \frac{a_n x^n}{n!} (1 - e^{-\frac{1}{b_n x^2}}) \qquad (1 < b_n > |a_n|)$$

has a_n as its nth derivative for $x = 0$, even if

$$\text{lim. sup. } \sqrt[n]{|a_n|/n!} = \infty.$$

This furnished the solution of a problem suggested by Professor Kasner, for which other solutions had been given by Borel and by Serge Bernstein.

In the present paper the domain of the variable is extended over a sector of the complex domain, and a solution of a problem proposed several years ago by Van Vleck is thereby obtained.

By a slight modification of the function above it is shown that a series of negative powers alone can be found which has a_n as its nth derivative for $x = 0$ relative to a sector of the plane.

An extension to the case of several variables is made.

2. In a former paper "On Babbage's functional equation," Mr. Ritt considered the cyclic groups of a class of functions in which the real linear fractional functions are included.

In the present paper it is shown that the only groups of functions of this class are the cyclic and the dihedral. It is shown also that any two isomorphic groups can be transformed into each other. The distribution of certain critical points connected with the group is studied. Applications are made to the theory of real linear transformations.

3. The neatest form of the equation of a rational plane curve of degree n, which we call the R^n, that may be derived from its parametric equations is found by equating to zero the n-rowed determinant obtained by equating to zero the Bezout eliminant of two line sections of the R^n, together with the application of a well-known translation scheme. The purpose of Professor Rowe's paper is to show how this same

equation may be derived from the equation of the line determined by two points of the R^n. This method involves neither the Bezout eliminant nor the translation scheme.

4. Take a finite number of points in a space of any number of dimensions. Let certain pairs of the points attract each other with a force proportional to the distance. The factors of proportionality for different pairs need not be the same. Let some of the points, called fixed, be held in arbitrarily assigned positions, while the others, called free, adjust themselves in positions of equilibrium. If each pair of points is connected by at least one chain of free points, each attracting the preceding and following, Professor Phillips calls the resulting configuration an elastic net. There is a unique position of equilibrium. The net has some curious properties. For example, if any free point is displaced from its equilibrium position, the force tending to return it is proportional to the displacement and independent of the positions of the fixed points. If any point, free or fixed, is moved, none of the free points can remain at rest, but all move in the same direction distances independent of the positions of the fixed points.

5. In 1901 Burali-Forti classified the singular points of analytic space curves not only with respect to the evenness or oddness of the exponents λ, μ, ν in the expansions

$$x = au^\lambda + \cdots, \qquad y = bu^\mu + \cdots, \qquad z = cu^\nu + \cdots,$$

but also with respect to the values of the curvature $1/r$ and the torsion $1/\rho$ of the curves at the singular points; and by combining the two principles he found fifty classes of such points. His classification, however, has the disadvantage of not being self-dual. In the present note Professor Ranum remedies this defect by taking into account the plane curvature $1/r'$ (defined and discussed by him in a recent number of the *Quarterly Journal*), which is the dual of the point curvature $1/r$. One result is that the number of classes is increased to seventy-four.

6. A postulate set for a deductive system is usually based on one or more undefined classes and one or more undefined relations, each relation being at least dyadic (operations are a special type of relation). By means of the notion of (1, 2)

correspondence Dr. Sheffer shows (1) how to reduce non-monadic relations to monadic; and, thus, (2) how to base postulate sets exclusively on (1, 2) correspondences and classes.

7. Postulate sets for groups, fields, Boolean algebras, and number algebras usually contain one or more existence postulates of the form

(α) There is a K-element x such that $x \circ x = x$,

or of the form

(β) There is a K-element x such that, for every K-element a,

$$a \circ x = x \circ a = a.$$

(Examples: $x =$ the addition modulus, 0; $x =$ the multiplication modulus, 1.) Postulates of type (α) or of type (β) may be called modular existence postulates. Dr. Sheffer shows how, in the construction of postulate sets, modular existence postulates may always be avoided.

8. A particular curve of the family of elliptic norm curves Q_n in S_{n-1} which admit a group G_{2n^2} of collineations is distinguished by a value of the parameter τ, itself an elliptic modular function defined by the modular group congruent to identity (mod n).

In the group G_{2n^2} there are certain involutorial collineations with two fixed spaces. If Q_n is projected from one upon the other, Q_n is mapped by a family of rational curves R_m with the parameter t. Under certain conditions the quadratic irrationality separating involutorial points on Q_n can define the elliptic parameter

$$u = \int \frac{(tdt)}{\sqrt{(t\tau)\alpha_\tau^{n-3}\alpha_t^3}}.$$

In Professor Miller's paper the form u is studied, is contrasted with Klein's form, and its natural occurence in connection with Q_3, Q_4, Q_5 is discussed.

9. In his well-known set of axioms for euclidean geometry Hilbert uses more undefined relations than are necessary for the construction of the desired geometric properties, i. e., on the basis of his system certain of his primitive relations may be defined in terms of others. The possible logical advantage

of such supernumerary indefinables, however, Hilbert does not bring clearly to view. The purpose of Mr. Schweitzer's note is to point out that supernumerary indefinables may sometimes be employed to eliminate explicit reference to existential properties in axioms; this is illustrated by his set of axioms for a field.*

10. Professor Fort's paper is divided into three parts. In the first part difference and differential equations are set up whose coefficients obey certain laws of which periodicity is a very special case and some fundamental theorems are proved for equations of this character.

Part two extends some familiar oscillation theorems primarily due to Sturm to equations of the type considered and proves the continuity of the characteristic values λ_j, for fixed boundary conditions and an interval of fixed length but of variable position, the characteristic values being considered as functions of the initial point of the interval.

Part three considers the so-called self-adjoint boundary value problems for the differential and difference equations where both coefficients depend upon a parameter λ. It is proved that the extremes of a certain type of functions λ_j considered under part two are the characteristic values for the problem in hand. The existence and number of these characteristic values is discussed, and means for distinguishing various cases, etc., taken up. A theorem of oscillation is given.

11. The general definition of adjoint boundary conditions associated with ordinary linear differential equations was given by Birkhoff (*Transactions*, 1908). Bôcher (*Transactions*, 1913) has investigated the circumstances under which a system of the second order is self-adjoint. In Dr. Jackson's paper a condition is given for self-adjoint sets of boundary conditions associated with differential equations of any order. The condition is expressed by a matrix equation of simple form, involving the given coefficients. It becomes particularly symmetric if the given differential equation itself is self-adjoint or anti-self-adjoint.

12. Professor Birkhoff begins by reducing the equations of motion for a dynamical system with two degrees of freedom

* Cf. BULLETIN, vol. 21 (1914–1915), p. 295.

to a special form which he has earlier employed.* In this way he is able to develop simple and general criteria for the existence of periodic orbits of a certain *primary* type. With the aid of these orbits a reduction of the dynamical problem to a problem in the transformation of surfaces into themselves is effected in a large class of cases. This reduction affords a means of proving the existence of infinitely many *secondary* periodic orbits, and of determining the precise structure of certain recurrent orbits and of orbits asymptotic to periodic and recurrent orbits. Finally the structure of the general orbit is determined. A number of applications of the theorems to special dynamical problems are given.

13. In a large part of the theory of functions of a single complex variable, the matrix of analytic functions rather than the single analytic function must be considered as the fundamental element. This is certainly the case for the functions defined by linear differential and difference equations.

The goal of the second paper by Professor Birkhoff is to show that the classical theorems of Weierstrass and Mittag-Leffler treating of the formation of functions with singularities of assigned type admit of a natural extension to matrices of analytic functions.

14. Professor Wilson calls attention to Ricci's generally neglected absolute calculus and to its suggestiveness as an implement of research in developing the theory of surfaces of two dimensions in euclidean space of n dimensions.

15. Professor Moore shows that the mean curvature of a two-dimensional surface in higher dimensions than three must be regarded as a vector magnitude and that the properties connected with the curvature of the surface may be expressed very simply with reference to an ellipse in the normal space, the vector from the surface point to the center of the ellipse being the mean curvature.

16. One of the two main outstanding problems in the theory of linear algebras is that of the invariantive classification of nilpotent algebras—that is, algebras such that some power of every number in the algebra is zero. In this paper Dr.

* *Rendiconti del Circolo Matematico di Palermo*, vol. 39, pp. 265–334.

Hazlett considers rational integral invariants of such algebras. All rational integral invariants for n-ary linear algebras under the total linear group reduce to zero for a nilpotent algebra, and accordingly we consider invariants under the group which leaves unaltered the canonical form. For such invariants this paper proves theorems analogous to theorems about invariants of algebraic forms, and in particular proves the finiteness of the rational integral invariants. The fundamental invariants are found for the simpler cases.

17. Vandiver and Fraenkel have studied finite algebras, but the former did not take up the question of factorization in such an algebra, while Fraenkel in the main restricts himself to decomposable (zerlegbar) rings or algebras. In this paper Dr. Kircher takes up the case of any finite algebra whose elements combine by addition and multiplication subject to the commutative, associative, and distributive laws, the algebra possessing unit elements with regard to both addition and multiplication. Division is not always possible, and when possible is not necessarily unique. Since every algebra fulfilling these conditions can always be represented by the residue classes of a modular system the subject is developed from this point of view. Since the law of unique factorization fails, a partial restoration is effected by the introduction of ideals among which a certain type defined as absolute prime ideals is of great use. These have the same relation to prime ideals in the algebra that an absolute prime modular system has to the irreducible modular system containing it. A number of theorems analogous to well-known number theory theorems are also obtained.

18. Professor Fréchet's paper appears in full in the present issue of the BULLETIN.

19. The groups discussed by Professor Kasner are related to the two types of conformal transformation of order two presented in a paper read at the Providence meeting and there called conformal symmetries and conformal involutions. The first type reverses the orientation of angles while the second preserves it. The groups are infinite, in the sense of involving an infinite number of parameters, and contain subgroups generated by operators of period two.

20. The principal theorem proved in Dr. Lipka's paper is a generalization of the geometric characterization of isogonal trajectories on a surface given by the author in the *Annals of Mathematics*, volume 15, No. 2. The generalized theorem states that if from the geodesic curvature of the ∞^3 geodesic curvature elements composing an isogonal family we subtract the geodesic curvature of the corresponding ∞^3 elements of *any* other isogonal family, and then rotate each element through a right angle, the ∞^3 new elements will form a natural family. This gives a general test for an isogonal family of curves on a surface.

21. In this paper Dr. Silverman establishes a correspondence between certain functions $f(z)$ and certain generalized definitions of the limit of a function of a continuous variable. The results obtained are similar to those presented to the Society in September, 1914, by Silverman and Hurwitz, who established a correspondence between certain functions $f(z)$ and certain definitions of summability of a divergent sequence. As in the case of sequences the following propositions are proved: (1) if $f(z)$ is analytic within and on the boundary of the circle C of radius $\frac{1}{2}$ about the point $\frac{1}{2}$, the corresponding definition is regular, i. e., correctly evaluates any existing limit of a function of a continuous variable; (2) all such definitions are consistent, i. e., if two of them furnish generalized limits of the same function, the values are the same; (3) two definitions are equivalent, i. e., have exactly the same scope of application, provided the corresponding functions have the same zeros with the same multiplicities in C; (4) the definitions for the limit of a function of a continuous variable, corresponding to those of Cesàro and Hölder of the same order for sequences, are equivalent. The last result is not new, having been first proved by Landau in 1913; but it appears here as a special case of (3).

22. One of the outstanding problems in the theory of surfaces is the determination of those surfaces which may be generated in two ways by the motion of invariable curves. The quadrics and surfaces of translation are well-known examples of such surfaces. Professor Eisenhart proposes and solves the problem of finding all ruled surfaces generated by the motion of an invariable curve whose points describe the

generators in the motion. It is found that cylinders and right conoids are the only surfaces possessing this property. A right conoid is a surface whose generators meet an axis to which they are perpendicular. A circular cylinder with the axis of the conoid for an element meets the conoid in a curve which goes into a congruent curve on the surface as the cylinder rolls on the envelope of the family of circular cylinders with the same radii, each having the axis of the conoid for an element.

23. Professor Eisenhart's second paper appeared in full in the January number of the *Transactions.*

24. Starting with any non-conjugate net N on a curved surface S, and a line g passing through each point of the surface and not lying in the tangent plane of that point, Dr. Green associates with the congruence Γ of lines g a second congruence Γ' bearing a certain characteristic relation R to the congruence Γ. To every point P of S and the line g through it, corresponds a line g' of the congruence Γ' lying in the tangent plane to S at P, and not passing through P. The relation R between the congruences Γ and Γ' is uniquely determined by the net of parameter curves N; if N is altered, the congruence Γ' is in general also changed. If N is a conjugate net, the relation R subsists between what Wilczynski* has called the axis and ray congruences. If N is not conjugate, the generalized axis and ray congruences as defined by Dr. Green in another paper† are also in the relation R. Probably the most remarkable case in which the relation R exists is that in which the net N is the asymptotic net, and the congruences Γ and Γ' are Wilczynski's directrix congruences. In this connection is given a new geometric characterization of these congruences: if N is asymptotic, then two congruences Γ and Γ' in the relation R have their developables in correspondence if and only if Γ and Γ' are the directrix congruences. Applications of the preceding ideas are made to the general theory of congruences and to the theory of surfaces.

25. Professor Osgood's paper gives a general definition of infinite regions by which ordinary complex n-dimensional space may be closed, projective space, the space of the geometry of

* *Transactions*, vol. 16 (July, 1915), pp. 311–317.
† Cf. the abstract in the July BULLETIN, vol. 21 (1914–15), pp. 484–5.

inversion, and the space of analysis being the most familiar examples. All such extended spaces are linearly simply connected. An extension of Weierstrass's theorem is then obtained, whereby a function which is meromorphic at every point of such a closed space is rational. It follows that any transformation of such a space into itself, which is regular at every point, is birational, and the Jacobian cannot vanish.

26. Professor Eiesland's paper is a continuation of the author's memoir in the *American Journal of Mathematics*, volume 35, entitled, "On a flat spread-sphere geometry in odd-dimensional space."

In the first part of the paper a method of obtaining surfaces with coordinate lines of curvature is given, based on two theorems which are proved. Surfaces whose lines of curvature are plane in all $n - 2$ systems have been derived and particularly the molding surfaces in S_{n-1} and the generalized Dupin cyclides of the third and fourth order.

The second part of the paper deals with the sphere-flat transformations, and surfaces with coordinate asymptotic lines.

The asymptotic lines on a sphere in $\overline{S}_{n-1}$ are obtained by the integration of a system of differential equations. This integration problem reduces to that of a Riccati equation and quadratures.

27. The usual geometric definition for the Plücker numbers of a plane curve, order, class, etc., is such as to require the recognition of both real and imaginary elements. In Professor Coolidge's paper new definitions are found which are suitable to the case where the universe of discourse includes only the real elements of the plane.

28. Professor Smith's paper shows that the trajectories described by a particle moving under central force varying inversely as the nth power of its distance from the center of force are completely characterized by two properties: (1) That the trajectories be isothermal; (2) That the isoclines be straight lines. By isoclines is meant the curves formed by joining points on consecutive curves which have parallel tangents.

29. If α is a λth root of unity, where λ is any integer, and

q a prime of the form $\lambda\nu + 1$, the Jacobi function $\psi(\alpha)$ has the property that $\psi(\alpha)\psi(\alpha^{-1}) = q$. Professor Mitchell discusses a more general function such that $\psi(\alpha)\psi(\alpha^{-1}) = q^t$, where q is any prime not contained in λ, and t any exponent for which $q^t \equiv 1$, mod. λ. For the case where λ is prime and t is the exponent to which q belongs, mod. λ, this function has been considered by Kummer. The present author determines the ideal factors of the function by essentially the same method which Kummer used in the special case mentioned. An error of Kummer's in this determination is found. Certain properties of these functions are also established.

30. The function $\psi(\alpha)$ considered in Professor Mitchell's first paper bears the same relation to the congruence $cx^\lambda + 1 \equiv dy^\lambda$ for the Galois field of order q^t that the Jacobi function does for the set of integral residues, mod. q, where q is a prime of the form $\lambda\nu + 1$. It is shown in his second paper that the number of solutions of any such congruence is determined if the functions $\psi(\alpha)$ are known. Two applications of this result are made, one to obtain an expression for the number of solutions of such a congruence in a field of order q^{st} in terms of those for a field of order q^t, and another to determine the exact values of these numbers for a field of order q^{2st}, provided $q^t \equiv -1$, mod. λ.

31. If $x_1, x_2, \cdots, x_n$ are n observed values, presumably equally accurate, of a magnitude which has an exact, but unknown, value, it is customary to regard the arithmetic mean of the observed values as the most probable value of the magnitude determined by them. A number of writers have attempted to justify this practice by selecting a set of properties which the most probable value ought to possess, and then proving that the set of properties characterizes the arithmetic mean. In his paper, Professor Beetle presents a set of three properties which characterizes the arithmetic mean, and also a set of three properties which characterizes the geometric mean.

If we denote the most probable value determined by $x_1, x_2, \cdots, x_n$ by $f_n(x_1, x_2, \cdots, x_n)$, the three properties which characterize the arithmetic mean are

(1) $f_n(x, x, x, \cdots, x) = x$;

(2) $f_n(x_1, x_2, \cdots, x_n)$ is a symmetric function of its arguments;

(3) $f_n(x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$

$$= f_n(x_1, x_2, \cdots, x_n) + f_n(y_1, y_2, \cdots, y_n).$$

The first two of the three properties characteristic of the geometric mean are the same as the first two for the arithmetic mean. The third is

(3′) $f_n(x_1y_1, x_2y_2, \cdots, x_ny_n)$

$$= f_n(x_1, x_2, \cdots, x_n)f_n(y_1, y_2, \cdots, y_n).$$

These sets of properties are not only extremely simple, but also exhibit very clearly the essential difference between the arithmetic mean and the geometric mean. In each set, the three properties are completely independent.

32. Concerning Hilbert's group-theoretic treatment of the foundations of geometry,* Poincaré says (according to Halsted's translation†): "Without doubt this is still not entirely satisfactory since though the *form* of the group is supposed any whatever, its *matter*, that is to say, the plane which undergoes the transformations, is still subjected to being a *number-manifold* in Lie's sense. Nevertheless, this is a step in advance, ..."

The treatment to which Poincaré here refers is based on three axioms (Axioms I, II and III). In an abstract‡ of a paper presented to the Society in April, Dr. Moore proposed a set of 12 axioms (Axioms 1–12) for plane analysis situs. These axioms are in terms of *point* and *region*. He desires to show that if to Axioms 1, 2, 4–7, 9–11 there be added the following Axioms A and B in addition to Hilbert's Axioms I, II and III (interpreted so as to apply properly in this new setting), then every space that satisfies the thus obtained set of axioms is either a euclidean or a Bolyai-Lobachevskian space of two-dimensions according as the group of all motions does or does not contain an invariant subgroup. This treatment (based on a set of axioms involving the notions point, region, and motion) does not presuppose that space, or even that any part of space, is a number manifold.

* D. Hilbert, "Ueber die Grundlagen der Geometrie," *Math. Annalen*, vol. 56 (1902).

† Cf. *Science*, May 19, 1911, p. 765.

‡ Cf. this Bulletin, vol. 21 (July, 1915), pp. 485–486.

A. If O is a point of a region R then there exists a region K containing O and such that K plus its boundary is a subset of R.

B. If A and B are distinct points and every region containing the point O contains a point of the point set M then there exists a point P belonging to M and such that every region containing A and B can be moved into a point set containing O and P.

33. Two curves corresponding point for point so that the tangents at corresponding points are parallel are said to be related by a transformation of Combescure. This transformation finds an analytic parallel in a second point-to-point correspondence of two curves, in which the tangents at corresponding points have as their common perpendicular direction that of the principal normal at the point of the given curve and make with one another an angle whose cosine is equal to the ratio of the corresponding elements of arc, so that the element of arc of the given curve is the projection of that of the transformed curve. The general and special correspondences of these two types are discussed in Dr. Graustein's paper.

34. The object of Mr. Gleason's paper is to establish Dirichlet's principle in n dimensions ($n \geq 2$) under very general conditions by a new method. The method consists partly in the formation of a function Υ from a function V, approximately satisfying the given boundary values, by assigning to Υ, as its value at any point of its region of definition, the weighted mean of V throughout an n-dimensional sphere with center at this point—the sphere being divided into concentric weighted spherical shells.

The n-dimensional region considered is bounded. The (inner) surface distribution v_s on its boundary S is required to be limited and continuous except possibly at a set of points (π) whose projected "area" on the coordinate hyperplanes is of outer content zero. Then (1) if the number of times a point P of S enters as a point of the inner boundary is finite, and either (2) the "projected area" of S on the coordinate hyperplanes is finite or (2′) every point is accessible in a certain sense from without, save at a set (π'), there exists a unique function, satisfying the given boundary values save at (π) and (π').

This harmonic function appears as the double limit of any sequences Υ for which the Dirichlet integrals of the corresponding set V, taken over any inner region, approach their lower bound for that region. The method applies also to infinite regions.

35. In the *Transactions*, volume 9, page 373, Professor Birkhoff has shown the general character of the expansion of an arbitrary function $f(x)$ in terms of the characteristic solutions of a certain linear differential system of the nth order, and has proved the convergence of the expansion. In the present paper Dr. Milne studies the degree of convergence of the same expansion, and shows that when $f(x)$ and its first $m - 1$ derivatives vanish at both ends of the interval the remainder after ν terms of the expansion will be less than a constant multiple of $1/\nu^m$ if $f^{(m)}(x)$ is continuous and of limited variation, and less than a constant multiple of $\log \nu/\nu^{m+1}$ if $f^{(m)}(x)$ satisfies a Lipschitz condition.

36. Professor Evans carries out, by means of curvilinear coordinates, a generalization to any curve of a method of Bôcher which was based on the circle. By means of a double integration, regarded as an iterated integration with respect to the curvilinear coordinates, formulas are obtained for solutions of Laplace's and Poisson's equations in terms of the boundary values on an arbitrary curve. The normal derivative of the Green's function thus appears as a ratio of two quantities expressible in terms of the curvilinear coordinates, and therefore can be calculated graphically whenever these can be drawn. Green's theorem is not used in the determination of the expressions for the solutions.

The method possesses the advantage that it applies directly to three dimensions as well as to two.

37. Mr. Alexander gives a simple proof that every Cremona plane transformation is the product of quadratic transformations. The paper will appear in the *Transactions*.

38. In the *American Journal of Mathematics* (1913) Professor Dines published a paper on the "Eliminant of several polynomials." At the suggestion of Professor Coble Mr. Robinson worked out the same problem by a direct method

analogous to the greatest common divisor process. His results were published in the last issue of the Johns Hopkins Circular (July, 1915).

F. N. Cole,
Secretary.

WINTER MEETING OF THE SOCIETY AT COLUMBUS.

The thirty-sixth regular meeting of the Chicago Section, being the fifth regular meeting of the American Mathematical Society in the west, was held at Columbus, Ohio, on Thursday, Friday and Saturday, December 30, 31, 1915, and January 1, 1916, in affiliation with the American Association for the Advancement of Science.

About one hundred persons were in attendance upon the various sessions, including the following sixty-seven members of the Society: Professor R. B. Allen, Professor Frederick Anderegg, Professor G. N. Armstrong, Professor R. P. Baker, Professor W. H. Bates, Professor P. P. Boyd, Professor Daniel Buchanan, Professor H. T. Burgess, Professor W. D. Cairns, Professor R. D. Carmichael, Professor H. E. Cobb, Professor Elizabeth B. Cowley, Dr. L. C. Cox, Professor D. R. Curtiss, Professor S. C. Davisson, Dr. W. W. Denton, Professor L. E. Dickson, Professor Peter Field, Professor B. F. Finkel, Professor T. M. Focke, Professor W. S. Franklin, Professor Harriet E. Glazier, Professor M. E. Graber, Professor Harris Hancock, Professor E. R. Hedrick, Dr. Cora B. Hennel, Dr. L. A. Hopkins, Professor L. C. Karpinski, Professor A. M. Kenyon, Mr. J. H. Kindle, Professor H. W. Kuhn, Professor Gertrude I. McCain, Dr. J. V. McKelvey, Dr. T. E. Mason, Professor F. E. Miller, Professor G. A. Miller, Professor J. A. Miller, Professor U. G. Mitchell, Professor C. N. Moore, Professor C. C. Morris, Professor F. R. Moulton, Professor A. D. Pitcher, Dr. V. C. Poor, Professor S. E. Rasor, Professor H. W. Reddick, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Professor R. E. Root, Professor D. A. Rothrock, Professor F. H. Safford, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Dr. H. M. Sheffer, Professor H. E. Slaught, Professor K. D. Swartzel, Dr. E. H. Taylor, Mr. C. E. Van Orstrand, Professor C. A. Waldo, Professor C. J. West, Professor H. S. White, Professor E. J. Wilczynski, Professor

F. B. Wiley, Professor C. B. Williams, Professor B. F. Yanney, Professor A. E. Young, Professor Alexander Ziwet.

The opening session on Thursday afternoon was a joint meeting with Section A of the American Association, at which the program consisted of

(1) Retiring address of Professor H. S. White, Vice-President of Section A, on "Poncelet polygons."

(2) Retiring address of Professor E. J. Wilczynski, Chairman of the Chicago Section, "Some remarks on the historical development and the future prospects of the differential geometry of plane curves."

(3) Address by Professor W. W. Campbell, of the Lick Observatory, President of the American Association, on "The rotation of planetary nebulæ."

In the absence of Professor A. O. Leuschner, vice-president of Section A, Professor C. A. Waldo was called upon to preside at this meeting. At the other sessions of the Chicago Section, the chairman, Professor E. J. Wilczynski, presided, with occasional relief by Professor H. S. White, former president of the Society.

At the business meeting of the Chicago Section on Friday afternoon, the following officers of the Section were elected for the ensuing period of two years: Professor W. B. Ford, chairman; Professor Arnold Dresden, Secretary; Professor H. L. Rietz, third member of the program committee. In presenting the nominations for these offices, Professor G. A. Miller, chairman of the committee, announced that Professor H. E. Slaught, who had served as Secretary of the Section since 1907, had expressed a desire to be relieved at this time. On Professor Miller's motion, appreciation of Professor Slaught's services was indicated by a unanimous rising vote.

Professor R. D. Carmichael reported for the retiring program committee that no arrangements had been made for a symposium at the next meeting and recommended that this question be continued in the hands of the new program committee. This recommendation was adopted.

The joint dinner of Section A and the Chicago Section on Thursday evening was held at the Ohio Union, where the arrangements and service were unusually satisfactory and highly appreciated. About seventy were present at the dinner, which was followed by an evening of enjoyable social intercourse, interspersed with some informal remarks by

Professors H. S. White, E. R. Hedrick, H. E. Slaught, and others who were called on by the chairman.

On Friday morning, the Department of Mathematics of Ohio State University tendered a luncheon to all mathematicians in attendance at the meetings, thus affording another opportunity for a social gathering and placing all guests of the occasion under renewed obligation to their hosts.

By a rising vote at the final session Saturday afternoon the members of the Society acknowledged their deep obligation to the local committee under the direction of Professor S. E. Rasor, to the Department of Mathematics of Ohio State University, and to the University as a whole, for the very careful attention which had been given to all the provisions for these meetings and for the generous hospitality which had been extended in every way throughout the period.

The following papers were presented at this meeting:

(1) Professor H. T. Burgess: "Note on the reduction of a family of quadratic forms."

(2) Professor J. B. Shaw: "Orthogonal vector systems in vector fields of three and more dimensions."

(3) Dr. A. J. Kempner: "On transcendental numbers."

(4) Dr. V. C. Poor: "A certain type of exact solutions of the equations of motion of a viscous liquid."

(5) Dr. V. C. Poor: "Transformation theorems in the theory of the linear vector function."

(6) Professor A. E. Young: "On the determination of a certain class of surfaces."

(7) Professor C. J. West: "Note on nine-fold and four-fold correlation."

(8) Professor R. D. Carmichael: "On a general class of series of the form $\Sigma\ C_n g(x + n)$."

(9) Dr. H. M. Sheffer: "On a set of independent postulates for complex algebra."

(10) Dr. H. M. Sheffer: "Mutually prime postulates."

(11) Professor C. H. Sisam: "On surfaces doubly generated by conics."

(12) Professor G. A. Bliss: "A note on the problem of Lagrange in the calculus of variations."

(13) Professor Daniel Buchanan: "Three-dimensional periodic orbits of a particle subject to the attraction of a sphere having prescribed motion."

(14) Dr. W. V. LOVITT: "A type of singular points for a transformation of three variables."

(15) Dr. W. W. KÜSTERMANN: "Functions of bounded variation."

(16) Professor T. H. HILDEBRANDT: "Green's functions connected with general linear differential equations."

(17) Professor C. N. MOORE: "On the developments in Bessel's functions."

(18) Professor E. J. WILCZYNSKI: "Integral invariants in projective geometry."

(19) Professor G. A. MILLER: "Limits of transitivity of a substitution group."

(20) Professor G. A. MILLER: "Finite groups represented by special matrices."

(21) Professor R. P. BAKER: "The four-color map theorem."

(22) Mr. W. L. HART: "Differential equations and implicit functions in infinitely many variables."

(23) Professor S. E. RASOR: "On the integration of Volterra's derivatives."

(24) Mr. A. R. SCHWEITZER: "An apparent anticipation of Hilbert's conception of completeness."

(25) Mr. A. R. SCHWEITZER: "A bifurcative generalization of a functional equation due to Cauchy."

(26) Miss IDA M. SCHOTTENFELS: "A class of functions which are self-reciprocal in the sense of Mellin."

Mr. Hart was introduced by Professors E. H. Moore and F. R. Moulton. The papers of Professors Shaw, Sisam, Bliss, and Hildebrandt, Drs. Lovitt and Küstermann, Miss Schottenfels, and the first paper of Mr. Schweitzer, were read by title.

Abstracts of the papers follow in the order indicated in the above list of titles:

1. Let $\lambda A + B$ be the matrix of a family of quadratic forms in which A is non-singular. Subject the family successively to the two linear transformations whose matrices are H and T respectively. Then

$$H'(\lambda A + B)H = H'AHH^{-1}(\lambda I + A^{-1}B)H = R(\lambda I + N),$$

$$T'R(\lambda I + N)T = T'RTT^{-1}(\lambda I + N)T = M(\lambda I + N).$$

Professor Burgess points out that the matrix H may be so

chosen that N is in the normal form and at the same time R is a matrix blocked off into principal minors each of which corresponds to an elementary divisor of $\lambda I + N$. The orders of the principal minors are the degrees of the corresponding elementary divisors; none of these principal minors overlap. He next shows that the matrix T may be so chosen that the normal form is unchanged and at the same time M differs from R in having all of its elements zero except those in the left principal diagonal of each of its blocks. As a consequence, all the well-known theorems on the elementary divisors of $\lambda A + B$ follow without further demonstration. The method of determining H and T is extremely simple and practical.

2. Professor Shaw's paper is a consideration of systems of three or more mutually orthogonal vectors in a field, such as the tangent, normal, and binormal of the vector lines, the normal and tangents of lines of curvature for a system of surfaces, and similar figures for polydimensional space. A linear vector operator of fundamental importance and much utility not hitherto noticed is exhibited, which for three-dimensional space is, in quaternion notation, α, β, γ being the moving orthogonal unit vectors,

$$\theta() = \rho() - (V\nabla\alpha)S\alpha - (V\nabla\beta)S\beta - (V\nabla\gamma)S\gamma,$$

where $2\rho = S\alpha\nabla\alpha + S\beta\nabla\beta + S\gamma\nabla\gamma$.

In terms of this operator the vector differential of any one of the three, in any one of the three directions is given by

$$(- S\lambda\nabla)\mu = V\mu\theta(\lambda).$$

The operator, of long-known use, $- S()\nabla \cdot \mu$, is given by

$$- S()\nabla \cdot \mu = V\mu\theta().$$

Application is made to congruences of lines.

3. Liouville (*Journal de Mathématiques*, volume 16, 1851) proved that $\sum_{r=0}^{r=\infty} \alpha_r/a^{m_r}$, where the α_r are real integers limited in absolute value and a a real integer $\geqq 2$, represents a transcendental number when the positive integral exponents $m_0, m_1, m_2, \cdots$ increase with sufficient rapidity. It is not

known, however, what constitutes this sufficient rapidity, so that the theorem does not serve to decide the transcendency of a number given in the form $\sum_{r=0}^{r=\infty}\alpha_r/a^{m_r}$ except in some extreme cases (including for example $m_r = r!, m_r = r^r$ (see Faber, *Mathematische Annalen*, volume 58, 1904).

Dr. Kempner proves that the power series

$$\sum_{r=0}^{\infty} \frac{\alpha_r}{a^{c^r}} \cdot x^r$$

has a transcendental value for every real rational value of x, if a and c are real integers $\geqq 2$, and if certain conditions are imposed on the integers α_r. These conditions are amply satisfied when: (1) $|\alpha_r| < M^r$, M arbitrary but fixed, and (2) only a finite number of the $\alpha_r = 0$.

4. The difficulty in obtaining exact solutions of the differential equations of a viscous liquid is due to their quadratic character. The most recent work of this kind has been done by C. W. Oseen. (See *Arkiv för Mathematik, Astronomi och Fysik*, volumes 3, 4, 6, 7, 9. Also *Acta Mathematica*, volume 34, 1911.) In the present paper, Dr. Poor proves the existence of a solution of the differential equations of a viscous liquid for all positive values of the time and in an infinite region. The body forces are assumed to be of the particular form

$$F(x, y, z, t) = F^{(1)}(x, y, z)e^{-k^2t}.$$

The method used is one of successive approximations. The successive steps involve solutions of sets of linear partial differential equations, the existence of which solutions is proved. Finally the convergence of the process is proved by using a dominance property of the successive steps. This dominance property persists if $F^{(1)}$ is properly restricted. Vector methods are used throughout the analysis.

5. Dr. Poor's second paper appeared in full in the January Bulletin.

6. In two previous papers, Professor Young has discussed the problem of determining various classes of surfaces, taking as the starting point the fundamental equations written in the form first suggested and used by Bonnet in similar work. In

the present paper, he discusses from the same standpoint the general class of surfaces characterized by having $D = \pm D''$, using the customary notation, the lines of reference being lines of curvature.

7. Statistical data in the social sciences frequently cannot be classified into more than two or three broad classes. Professor West derives the working formulas for the application of the method of the correlation ratio to this type of correlation problems and discusses the value of the method as compared with certain other methods that have been proposed.

8. The series treated by Professor Carmichael are of the form

$$\Omega(x) = \sum_{n=0}^{\infty} c_n g(x+n), \qquad \bar{\Omega}(x) = \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)},$$

where $g(x)$ is a function having the asymptotic character

$$g(x) \sim x^{P(x)} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right)$$

valid for x approaching infinity in a positive sense along any line whatever parallel to the axis of reals, the functions $P(x)$ and $Q(x)$ being polynomials. For the special case in which $g(x) = 1/\Gamma(x)$ the series $(\bar{\Omega}x)$ is a factorial series. Professor Carmichael points out that the series $\Omega(x)$ and $\bar{\Omega}(x)$ are of great importance in investigating the properties of functions in the neighborhood of singularities of certain frequently occurring sorts and indicates his purpose to devote several memoirs to the development of a general theory of these series, especially in relation to the function-theoretic problem mentioned. In the present paper the foundation of a general theory of these series is laid. In case the series $\Omega(x)$ neither converges everywhere nor diverges everywhere its region of convergence [absolute convergence] is the half-plane $R(\sigma x) \leqq \lambda$ $[R(\sigma x) \leqq \mu]$, where $\lambda[\mu]$ is a constant depending on $c_0, c_1, c_2, \cdots$ and σ is the coefficient of the leading term in $Q(x)$ or in $P(x)$ according as $Q(x)$ is or is not of greater degree than $P(x)$. For the values of λ and μ explicit formulas are given analogous to the Cauchy-Hadamard formula for the radius of convergence of a power series. The paper contains

also a treatment of uniform convergence, of the existence of singularities of the sum-function on the boundary of the region of convergence and of uniqueness of expansions in these series. A given function $f(x)$ has not more than one expansion $\Omega(x)$ or $\overline{\Omega}(x)$ when $g(x)$ is given and $R(\sigma) > 0$. For use in investigating this theory it was found convenient to employ a certain interesting generalization of the "generalized Dirichlet series"; and the theory of these generalized series was developed to the extent needed for the applications in question.

9. The postulate set for ordinary complex algebra presented by Dr. Sheffer is based on an undefined class and three undefined operations. The set differs from previous sets in that: (1) the number of undefined entities and of postulates is reduced; (2) the order relation is defined; and (3) the existence of 0, 1, negatives, reciprocals, and imaginaries is proved.

10. A set of m postulates such that no $m - 1$ of the postulates imply the mth is called independent. If P and Q are any two postulates of an independent set, P does not imply the whole of Q. P may imply, however, a part of Q. Two postulates, neither of which implies any part of the other, may be called mutually prime; and a set of postulates which are mutually prime by pairs may be called a set of mutually prime postulates. Obviously, mutual primeness implies independence; but not conversely. Also, mutual primeness implies E. H. Moore's complete independence; but not conversely. Dr. Sheffer shows how to construct sets of postulates which are mutually prime.

11. In this paper, Professor Sisam determines some fundamental properties of the surfaces of order seven and eight which contain two pencils of conics.

12. Professor Bliss's paper appeared in full in the February BULLETIN.

13. In an article entitled "A class of periodic orbits of an infinitesimal body subject to the attraction of n finite bodies" (*Transactions*, volume 8 (1907), pages 159–188), Longley discussed the periodic motion of a particle which moves subject to the Newtonian attraction of n finite bodies having pre-

scribed motion. The finite bodies and the particle are restricted to move in the one plane and the coordinates of the finite bodies, when referred to one of the bodies as origin, are assumed to be known functions of the time. In the present paper Professor Buchanan discusses the periodic motion of the particle when the finite bodies have the motion prescribed in Longley's article, but the particle is not restricted to move in a plane. Periodic solutions are determined as power series in a certain parameter which may be expressed as a function of the initial projection from the plane of motion of the finite bodies. These solutions have a period commensurable with the period of Longley's solutions and reduce to the latter when the initial projection from the plane of motion becomes zero.

14. Dr. Lovitt's paper appeared in full in the February BULLETIN.

15. The idea of a function of bounded variation, first developed for functions of one variable by Jordan, has been extended to two variables by Arzelà and Hardy in papers published in 1905. These authors' generalizations differ definitionally. Dr. Küstermann asks whether they coincide conceptually and shows that they do not by constructing a function which, while monotonically increasing in both x and y, and hence of bounded variation in Arzelà's sense, is not so according to Hardy's definition. Since both types of functions are integrable and can be developed into a double Fourier series it thus appears that Arzelà's generalization is not only the broader, but also the more natural one, preserving more closely the analogy to functions of a single variable. Nevertheless, in recent papers, Lebesgue and W. H. Young, apparently unacquainted with Arzelà's work, are using Hardy's definition. At any rate the designation "function of bounded variation in two variables" is not unique, but applies to-day to two distinctly different classes of functions.

16. The paper of Professor Hildebrandt is a generalization, in the sense of Moore's general analysis, of the memoir by Schlesinger: "Zur Theorie der linearen Integralgleichungen"*

* *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 24, pp. 84–123.

and that of Bounitzky: "Sur la fonction de Green des équations différentielles linéares ordinaires."* The differential equations treated are of the form:

$$(1) \quad \frac{d}{dx}\eta(p'p''x) - \alpha(p'p''x) - J_{q''q'}\alpha(p'q''x)\eta(q'p''x) = 0,$$

$$(2) \quad \frac{d}{dx}\eta(p'p''x) - J_{q''q'}\alpha(p'q''x)\eta(q'p''x) = 0,$$

where p' and p'' range over two general classes $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively, x ranges over a bounded linear interval $x_0 \leqq x \leqq x_1$, α and η are suitably conditioned functions of these variables, and J is a linear operator on functions of the type $\kappa(q''q')$. The first part of the paper contains existence theorems, general solutions of the equations (1) and (2), a consideration of the analogue of the Fredholm determinant for the solutions, and a Green's theorem relating to the right-hand members of equations (1) and (2), and suitably defined adjoint expressions. The second part is devoted to the consideration of a system of Green's functions $\Gamma(p'p''x, y)$ which are continuous in x and y except for $x = y$, where the discontinuity is a function of the form $\kappa(p'p'')$. These Green's functions satisfy equations of the form (2), and certain general boundary conditions. The usual theorems relating to such functions are obtained, as well as the solutions of non-homogeneous equations of the form

$$\frac{d}{dx}\eta(p'p''x) = \alpha_0(p'p''x) + J_{q''q'}\alpha(p'q''x)\eta(q'p''x),$$

satisfying the given initial conditions.

17. There are certain methods of establishing the convergence of the developments in Bessel's functions that leave unsettled the question of the value to which they converge. One of these methods is the only means thus far available of establishing the convergence at the origin of the developments in Bessel's functions of order zero for the case where the function developed has discontinuities.† Hence, in order to have a complete treatment of the developments in question

* *Journal de Mathématiques*, ser. 6, vol. 5 (1909), pp. 65–125.
† Cf. *Transactions*, vol. 12 (1911), p. 181.

that is adequate for the applications to mathematical physics and at the same time is not unnecessarily cumbersome, it is desirable to have a simple proof of the fact that the convergence is to the desired value. In Professor Moore's paper such a proof is given.

18. The simplest integral invariant of the projective theory of plane curves may be written in the form

$$(1) \qquad H = \int \sqrt[3]{\theta_3}\, dx,$$

where

$$(2) \qquad y''' + 3p_1y'' + 3p_2y' + p_3y = 0$$

is the differential equation of the curve, and where

$$P_2 = p_2 - p_1^2 - p_1', \qquad P_3 = p_3 - 3p_1p_2 + 2p_1^3 - p_1'',$$
$$\theta_3 = P_3 - \tfrac{3}{2}P_2'.$$

Any other integral invariant of the curve may be expressed in the form $\int IdH$, where I is any one of its absolute differential invariants.

The integral H is as fundamental in the projective theory of plane curves as the length of arc, the fundamental integral invariant of the metric theory, is for metric geometry. The purpose of Professor Wilczynski's paper is to call attention to this fact and to provide a geometric interpretation for the integral H. His interpretation is as follows:

Consider any arc of the curve corresponding to the interval $a \leqq x \leqq b$ of the independent variable. Divide this interval into n parts by means of the values $x_0 = a, x_1, x_2, \cdots x_{n-1}, x_n = b$ such that $\lim \delta x_k = \lim (x_{k+1} - x_k) = 0$ as n grows beyond bound. Let $A, P_1, P_2, \cdots P_{n-1}, B$ be the points on the curve which correspond to these $n + 1$ values of x. Let t_k be the tangent and C_k the eight-pointic nodal cubic of P_k. The three points of inflection of the cubic C_k are on a line i_k which intersects t_k in a point I_k. Denote by τ_k one of the inflectional tangents of C_k and let T_k be its intersection with t_k. The line P_kP_{k+1} will intersect i_k and t_k in two points I_k' and T_k', and the cross-ratio $(I_k', T_k', P_k, P_{k+1})$ turns out to be equal to

$$1 - \frac{3}{\sqrt[3]{10}} \sqrt[3]{\theta_3(x_k)}\, \delta x_k.$$

By a perspective correspondence the three points I'_{n-1}, T'_{n-1}, P_{n-1} of $P_{n-1}B$ may be projected into the points I'_{n-2}, T'_{n-2}, P_{n-1} of $P_{n-2}P_{n-1}$. Let B_{n-1} be the point of $P_{n-2}P_{n-1}$ which in this perspective corresponds to B. Then project similarly I'_{n-2}, T'_{n-2}, P_{n-2}, B_{n-1} into the four points I'_{n-3}, T'_{n-3}, P_{n-2}, B_{n-2} of $P_{n-3}P_{n-2}$, and continue in this way. We shall finally obtain upon the line AP_1 a point B_1 determined from B by this sequence of perspectives. As n grows beyond bound, B_1 will approach a limiting position Q on the initial tangent t_0 of the arc AB. Let κ denote the double-ratio $\kappa = (I_0, T_0, A, Q)$. We shall have

$$\log \kappa = \frac{3}{\sqrt[3]{10}} \int_a^b \sqrt{\theta_3}\, dx,$$

which gives the desired interpretation.

It is also easy to write down projective integral invariants in the theory of space curves and surfaces. None of these however have as yet received any interpretation.

19. On page 68 of this volume of the BULLETIN Professor Miller established the theorem that a substitution group of degree n which is neither alternating nor symmetric cannot be more than $3\sqrt{n} - 2$ times transitive when $n > 12$. In the present note he points out that it results from the main theorem proved in the article mentioned that such a group cannot be more than $\frac{5}{2}\sqrt{n} - 1$ times transitive. For large values of n this evidently gives a much smaller upper limit for the degree of transitivity than the one mentioned above, which is itself much smaller than the one commonly given, namely, $\frac{1}{3}n + 1$.

20. The direct object of Professor Miller's second paper is to prove that every finite group which contains an abelian subgroup of half its order can be represented by square matrices all of whose elements are equal to zero, with the exception of those which appear in one of the diagonals, and all of these are ordinary complex numbers which are different from zero. Incidentally several other somewhat general theorems are established. Among these are the following:

If G is an abelian group of order p^m and if H is a subgroup of order p^α, then a set of independent generators of G can

always be so selected that at most $m - \alpha$ of them are not contained in H whenever G contains more than $m - \alpha$ independent generators. Moreover, it is always possible to construct a group having k independent generators and an arbitrary quotient group of order p^α such that the subgroup corresponding to identity of this quotient group cannot involve more than $k - \alpha$ of the operators in any possible set of independent generators of the group. If t transforms an abelian group G of order 2^m according to an automorphism of order 2 and if H is the subgroup of G formed by its operators which are invariant under t, then a set of independent generators of G can be so chosen that H contains all of them with the exception of at most three for each invariant of G/H.

21. The problem of the four-color map is taken by Professor Baker in the dual form. Every polyhedral net on the sphere can have its vertices marked with four colors so that no edge has the same color at its ends. The net is prepared and the problem reduced to that of an all-triangle polyhedron without triple circuits. This class is shown to be traversable by a closed curve passing once through all the vertices and having any pair of connected edges on the contour. For such a configuration a cardinal number relation is obtained for the number of successful colorings which is used as the basis of a two-step descending induction. The corresponding problem for one-sided closed surfaces of connectivity 2, 3, 4 is also solved. The numbers are 6, 6, 7.

22. At the April (1915) meeting of the Chicago Section, Mr. Hart presented a preliminary report which dealt with a part of the results of his present paper.

This investigation is concerned with functions f of the real variable $\xi = (x_1, x_2, \cdots)$ in the space

$$R\colon\quad M_1^{(i)} \leqq x_i \leqq M_2^{(i)} \qquad (M_1^{(i)}, M_2^{(i)} \leqq M;\ i = 1, 2, \cdots).$$

A function f is said to be completely continuous at the point ξ of R if, whenever

$$\lim_{n=\infty} x_{in} = x_i \qquad (i = 1, 2, \cdots),$$

it follows that

$$\lim f(\xi_n) = f(\xi) \qquad (\xi_n = x_{1n}, x_{2n}, \cdots).$$

The results obtained are of three sorts. In the first place, theorems on completely continuous functions are derived which include, for example, the Weierstrass theorem on uniformly convergent sequences of continuous functions and Taylor's theorem. In a second part of the paper the fundamental theorem of implicit function theory is proved for the infinite system of equations

$$f_i(\xi, \eta) = 0 \quad [\xi \text{ in } R;\ \eta = (y_1, y_2, \cdots);\ \eta \text{ in } R], \tag{1}$$

which defines ξ as a function of $(y_1, y_2, \cdots)$. Then, finally, there is considered the infinite system of ordinary differential equations

$$\frac{dx_i}{dt} = f_i(\xi, t) \tag{2}$$

$$(i = 1, 2, \cdots;\ x_i(t_0) = a_i;\ \xi \text{ in } R;\ |t - t_0| \leqq r_0),$$

and, under suitable hypotheses, the existence of a unique continuous solution $\xi(t)$ is established.

In the second and third parts of the paper the existence proofs are constructed by methods of successive approximation in which the theorems of the first section are of fundamental importance. The classical existence theorems for finite systems of implicit functions and differential equations are special cases of the results for systems (1) and (2).

23. In the *Rendiconti dei Lincei*, Volterra defined functions of lines, their continuity, and their derivatives. In later publications, he gave a method by means of Stokes' theorem for finding anti-derivatives for these functions of a line. The object of Professor Rasor's paper is to point out an instance quite analogous to the above from the calculus of variations, using Euler's equation for this purpose.

24. The object of Mr. Schweitzer's note is to call attention to Kempe's "law of continuity" which he phrases* as follows: "No entity is absent which can consistently be present." Kempe is careful to remark that the function of this law is to ensure the "complete definition" of his system (l. c., page 149) and that the law applies to "geometric sets" (page 177).

* *Proceedings London Math. Soc.*, vol. 21.

It appears* that Kempe's statement may desirably replace Hilbert's "axiom of completeness." Both principles emphasize the interdependence of mathematics and psychology and the problem presented by the relativity of the principles as used by their respective authors seems worthy of careful study.

25. Equivalent to Cauchy's well-known functional equation $\lambda(x + y) = \lambda(x) + \lambda(y)$ are the equations

$$\lambda(x - z) - \lambda(x - y) = \lambda(y - z), \tag{1}$$

$$\begin{aligned} \lambda(z) + \lambda(x + y) &= \lambda(x) + \lambda(y + z) \\ &= \lambda(y) + \lambda(x + z), \quad \lambda(0) = 0. \end{aligned} \tag{2}$$

In Mr. Schweitzer's second paper an interesting "bifurcation" is represented by the following generalizations:

$$\begin{aligned} &f\{\lambda_1 f(t_1, t_2, \cdots, t_n, x_1), \cdots, \lambda_{n+1} f(t_1, t_2, \cdots, t_n, x_{n+1})\} \\ &\qquad = \mu f(x_{i_1}, x_{i_2}, \cdots, x_{i_{n+1}}), \quad i_k = 1, 2, \cdots, (n + 1), \end{aligned} \tag{1'}$$

$$\begin{aligned} &\mu_i \phi\{\lambda_1(x_1), \cdots, \lambda_j(x_{i-1}), \lambda_{j+1}(x_{i+1}), \cdots, \lambda_n(x_{n+1}), \\ &\qquad \lambda_{n+1}\phi(x_i, t_1, t_2, \cdots, t_n)\} \\ &= \phi\{\lambda_1(t_1), \lambda_2(t_2), \cdots, \lambda_n(t_n), \lambda_{n+1}\phi(x_1, x_2, \cdots, x_{n+1})\} \end{aligned} \tag{2'}$$

where in the latter† system $i = 1, 2, 3, \cdots, (n + 1)$, $x_0 \equiv x_2$, $x_{n+2} \equiv x_n$. Equations (1′) have been previously discussed by the author. In the case of equations (2′), the specialization $\lambda_1(x) = \lambda_2(x) = \cdots = \lambda_{n+1}(x) = \mu_i(x) = x$ leads to functional equations discussed by Abel, Stäckel, and Hayashi.

26. In the *Mathematische Annalen*, volume 68 (1910), pages 314–326, Mellin states the two following reciprocity theorems:

I. If $F(x)$ is any function belonging to a properly defined class and if

$$\varphi(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(x) t^{-x} dx,$$

* Cf. the "axiom of completeness" of G. Rabinovitch, BULLETIN, vol. 12 (1905–1906), p. 433 (abstract): "no motion is impossible unless it contradicts the above axioms."

† When $i = n + 1$, the argument of λ_n is x_n.

then

$$F(x) = \int_0^\infty \varphi(t) t^{x-1} dt.$$

II. If $\varphi(x)$ satisfies the conditions in Theorem I, and if $F(t) = \int_0^\infty \varphi(x) x^{t-1} dx$, then reciprocally,

$$\varphi(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(t) x^{-t} dt.$$

Example:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt,$$

$$e^{-t} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(x) t^{-x} dx \text{ for } a > 0; \ -\frac{\pi}{2} < \text{arg. } t < \frac{\pi}{2}.$$

Miss Schottenfels' paper treats of a class of functions which are self-reciprocal in the above sense of reciprocity.

H. E. Slaught,
Secretary of the Chicago Section.

ON PIERPONT'S DEFINITION OF INTEGRALS.

BY PROFESSOR M. FRÉCHET.

(Read before the American Mathematical Society, December 27, 1915.)

In the second volume of his Lectures on the Theory of Functions of Real Variables, Professor J. Pierpont has given a new definition of Lebesgue integrals. This definition is interesting in as much as it realizes an effort to adapt the previous methods of presentation of Riemann integrals to the newer Lebesgue integrals.

But unfortunately the happiness of this idea is lessened in Pierpont's work by the choice of an inappropriate definition. Professor Pierpont intended to generalize the definition of Lebesgue integrals by defining upper and lower integrals of any function $f(x)$ on any linear set E_μ. Such definitions should not, of course, be arbitrary ones, and there are some primary conditions to be fulfilled, unless these definitions are to become quite artificial and uninteresting.

For instance, it is to be expected that for any $f(x)$ and any E,

$$\overline{\int_E} f \geqq \underline{\int_E} f,$$

if the left and right sides denote the upper and lower integrals of $f(x)$ over E. And when $f \equiv 1$ these integrals ought to reduce to the upper and lower measures of E. However, if $f \equiv 1$ it will be found that those integrals are respectively the lower and upper bounds of $\Sigma\, \overline{\text{meas.}}\, \delta_n$, where $\delta_1, \delta_2, \cdots, \delta_n, \cdots$ is a "separated division of A into cells."* Hence

$$\overline{\int_E} f \leqq \underline{\int_E} f;$$

and the equality cannot hold for every E. For if, for instance, E is the interval (0, 1) and if δ_1 is a non-measurable part of E and $\delta_2 = E - \delta_1$, then $\overline{\text{meas.}}\, \delta_1 + \overline{\text{meas.}}\, \delta_2 > \text{meas. } E$. But of all the separated divisions of E, take those two the first of which consists of E itself and the second of (δ_1, δ_2); then

$$\overline{\int_E} f \leqq \text{meas. } E,$$

$$\underline{\int_E} f \geqq \overline{\text{meas.}}\, \delta_1 + \overline{\text{meas.}}\, \delta_2.$$

Thus we get a case where at the same time

$$\overline{\int_E} f < \underline{\int_E} f$$

and, though $f = 1$,

$$\underline{\int_E} f > \text{meas. } E.$$

Curiously enough, Professor Pierpont did not think it useful to mention that the inequality

$$\overline{\int_E} f - \underline{\int_E} f \geqq 0$$

---

* δ_i, δ_k are said to be separated when they are enclosed respectively in two measurable sets whose common part has the measure zero. Here $\overline{\text{meas.}}$ means upper measure.

should hold, as should result from his relation

$$\overline{\int_E} f - \underline{\int_E} f = \lim. \Sigma\omega_n \times \overline{\text{meas.}}\ \delta_n,$$

where ω_n is the oscillation of f in δ_n. However, it results from my example above that this relation is not always true.

The mainspring of all these difficulties is the error made in theorem 376, page 369: "Let $A = (B, C)$ be a separated division of A, then $\overline{\text{meas.}}\ A = \overline{\text{meas.}}\ B + \overline{\text{meas.}}\ C$," which is a generalization of theorem 341, page 346, "If $A = B + C$ and B, C are exterior to each other, $\overline{\text{meas.}}\ A = \overline{\text{meas.}}\ B + \overline{\text{meas.}}\ C$." The assumption made in the second line of the proof of this theorem is not altogether obvious, so that the proof is not convincing. Moreover, the theorem itself does not hold in every case; for instance, it does not when A is an interval and B a non-measurable part of A.

It is further found that the demonstration of the inequalities

$$(1) \qquad m \times \overline{\text{meas.}}\ E \leqq \underline{\int_E} f; \qquad \overline{\int_E} f \leqq M \times \overline{\text{meas.}}\ E$$

is based explicitly (§ 379, page 372) on the first theorem reproduced above and on a consequence of it which reads as follows:

$$m \times \overline{\text{meas.}}\ E \leqq \underline{S}_D; \quad \overline{S}_D \leqq M \times \overline{\text{meas.}}\ E.$$

Now this consequence is easily seen to be false itself, whereas the final inequalities (1), which are correct, would have been more easily proved by showing that $m \times \overline{\text{meas.}}\ E$ and $M \times \overline{\text{meas.}}\ E$ are particular values assumed by $\underline{S}_D$ and $\overline{S}_D$ when D consists of E alone.

At any rate, many difficulties should disappear if the δ_n are to be measurable. No doubt E would then itself be measurable and the definition would not have so large an extent. However the case of the non-measurable E—which is not particularly interesting—may be easily dealt with by enclosing E in any measurable set B, letting $f = 0$ in $B - E$ and putting

$$\underline{\int_E} f = \underline{\int_B} f, \qquad \overline{\int_E} f = \overline{\int_B} f,$$

these values being obviously independent of the choice of B.

Finally, I fail to see any advantage in the use of the so-called separated divisions of E. The results are exactly the same and a useless complication is avoided if E is only divided into parts exterior to each other.

Now divide a measurable set E into a countable sequence of measurable subsets δ_i exterior to each other and denote by

$\overline{\int_E} f$ and $\underline{\int_E} f$ the lower and upper bounds of $\Sigma M_i \delta_i$ and $\Sigma m_i \delta_i$,

where m_i, M_i are the lower and upper bounds of f on δ_i. By these definitions, the upper integral is never smaller than the lower integral. And if $f \equiv 1$, both integrals are equal to meas. E.

This new definition is very similar to that of Riemann. The real difference is *not* as Professor Pierpont asserts for his own that it makes use of an infinite instead of a finite number of parts of E. It lies essentially in the use of measurable parts of E instead of intervals. For instance when f is bounded over E, the definition is not altered if the parts δ_i of E are assumed to be in infinite (variable) number.

When $\underline{\int_E} f = \overline{\int_E} f$ the common value of both integrals is equal to the value of the corresponding Lebesgue integral.

UNIVERSITY OF POITIERS.

REPLY TO PROFESSOR FRÉCHET'S ARTICLE.

1. REPLYING to the foregoing criticism I begin by quoting. Professor Fréchet says: "But unfortunately the happiness of this idea is lessened in Pierpont's work by the choice of an inappropriate definition. Professor Pierpont intended to generalize the definition of Lebesgue integrals by defining upper and lower integrals of any function on any linear set E_μ. Such definitions should not of course be arbitrary ones, and there are some primary conditions to be fulfilled unless these definitions are to become quite artificial and uninteresting."

The implication that the reader will easily draw is that I have not fulfilled these primary conditions and that my theory is therefore quite artificial and uninteresting. Certainly flattering to the author.

To be historically accurate, I had no intention whatever of generalizing Lebesgue's integrals. When years ago I hit on my definition of integration, I did not know how it was related to Lebesgue's theory. I found out later that when the field of integration is measurable my integrals are identical with Lebesgue's and I have therefore called them Lebesgue integrals throughout my book. To prevent misunderstanding let me note that my definition is not restricted to a single variable as one may have gathered from the passage just quoted; this however is a minor matter. Professor Fréchet calls my definition inappropriate. Since my integral and Lebesgue's are the same when the field of integration is measurable, any defect in my integral is equally shared by Lebesgue's in this case. I infer therefore that his strictures apply only to the case where the field of integration is non-measurable.

I lay no great importance on this side of my definition. No non-measurable field has yet been studied as far as I know, and it may turn out that they have little value in the theory of point sets.

Theoretically they do present themselves in a rather awkward way in the theory of double integrals. Let $\mathfrak{A}$ be a measurable limited field whose projection is $\mathfrak{B}$ and whose cross sections are $\mathfrak{C}$. If $f(x, y)$ is limited and integrable in $\mathfrak{A}$, we would like to write down, as in the calculus,

$$\int_{\mathfrak{A}} f(x, y)dxdy = \int_{\mathfrak{B}} dx \int_{\mathfrak{C}} f(x, y)dy. \tag{1}$$

Now it turns out that although $\mathfrak{A}$ itself is measurable, an infinite number of the sections $\mathfrak{C}$ *may not be*. If now we do not define integrals over non-measurable fields, the symbol

$$\int_{\mathfrak{C}} f(x, y)dy$$

which enters (1) is not defined and the same is true of the right side of (1). This difficulty may be turned in a variety of ways; one way is to use a definition of integration which does not depend on the measurability of the field.

This Professor W. A. Wilson did in 1909. He replaced the intricate and highly artificial reasoning of Lebesgue* by

* Cf. Lebesgue, *Ann. di Mat.*, 1902. Reproduced by Hobson, Functions of a Real Variable (1907), p. 576.

simpler and more direct methods as given in Volume II of my Real Variables.*

2. Let us now consider some of Professor Fréchet's objections in detail, it being understood that the field of integration is *non*-measurable. He says: It is to be expected that for any $f(x)$ and any E

$$\overline{\int_E} f \geqq \underline{\int_E} f. \tag{2}$$

Now Professor Fréchet thinks he has constructed an example which contradicts the relation (2), i. e., he thinks he has shown that in a certain case

$$\overline{\int_E} f < \underline{\int_E} f. \tag{3}$$

If this were true, my theory of integration for non-measurable fields would be in a sorry plight.

To establish the *false* relation (3) for a special case, Professor Fréchet divides the unit interval $E = (0, 1)$ into two parts by taking a non-measurable component δ_1 and its complement δ_2 such that

$$\overline{\text{meas}}\ \delta_1 + \overline{\text{meas}}\ \delta_2 > \text{meas}\ E. \tag{4}$$

So far so good, but Professor Fréchet now states that $\mathfrak{A} = (\delta_1, \delta_2)$ is a *separated* division, for he says: "But of all separate divisions of E take those two, the first of which consists of E itself and the second of (δ_1, δ_2)."

Professor Fréchet has been misled at this point; there is no separated division of $\mathfrak{A}$ such that (4) holds, and his example establishes not an error on my part but a carelessness of reasoning on his.

3. Professor Fréchet now attacks the correctness of the relation

$$\overline{\text{meas}}\ A = \overline{\text{meas}}\ B + \overline{\text{meas}}\ C, \tag{5}$$

where B, C is a separated division of $\mathfrak{A}$. He says:

(1) "The assumption made in the second line of the proof of this theorem is not altogether obvious, so that the proof is not convincing. (2) Moreover, the theorem itself does not

* Wilson's results were given by me in a course of two lectures delivered at Clark University in September, 1909. Cf. also a paper by Hobson, *Proceedings Lond. Math. Soc.*, vol. 8, Part 1. Issued December 23, 1909.

hold in every case; for instance, it does not when A is an interval and B is a non-measurable part of A."

As far as the writer can see, (2) is a bald statement unaccompanied by a shred of proof. A charge as serious as this certainly deserves some support on the part of the person making it.

Let us look at (1). The assumption in question is that

$$\mathfrak{E}_n = A_n + B_n + C_n \tag{6}$$

is an ϵ_n-enclosure of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ simultaneously.

The author made this statement without proof because in his judgment an attentive reader who had used the machinery of superposition up to this point would admit its truth as obvious. However to make it clear even to him that runs, we add the following, using the notation of my book:

Let

$$A_n = \{a_{n\kappa}\}, \quad B_n = \{b_{n\lambda}\}, \quad C_n = \{c_{n\mu}\}.$$

Each cell $a_{n\kappa}$ is a measurable point set containing points of $\mathfrak{A}$, $b_{n\lambda}$ one containing points of $\mathfrak{B}$, etc. Now if e_{nj} denote a cell of $\mathfrak{E}_n$, e_{nj} is by definition $Dv\{a_{n\kappa}, b_{n\lambda}\}$ or $Dv\{a_{n\kappa}, c_{n\mu}\}$. But any point of $\mathfrak{B}$ (or of $\mathfrak{C}$) is in some $b_{n\lambda}$ (or $c_{n\mu}$) and since $\mathfrak{B}$ (or $\mathfrak{C}$) is a part of $\mathfrak{A}$, in some $a_{n\kappa}$. Therefore $\mathfrak{E}_n = \{e_{nj}\}$ contains all points of $\mathfrak{A}$. Now $\mathfrak{E}_n$ is a part of A_n and hence meas $\mathfrak{E}_n \leqq$ meas A_n; therefore $\mathfrak{E}_n$ is an ϵ_n enclosure of $\mathfrak{A}$. Let B_n' denote those parts of B_n contained in $\mathfrak{E}_n$. Then meas $B_n' \leqq$ meas B_n. Hence B_n' is an ϵ_n enclosure of $\mathfrak{B}$. Similarly C_n' is an ϵ_n enclosure of $\mathfrak{C}$. Thus $\mathfrak{E}_n$ is simultaneously an ϵ_n enclosure of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$.

4. The next charge is that the relations

$$m\overline{\overline{\mathfrak{A}}} \leqq \underline{S}_D, \quad \overline{S}_D \leqq M\overline{\overline{\mathfrak{A}}} \tag{7}$$

are false. The proof that they are *correct* is given on page 372; as it requires but four lines, I reproduce it textually, viz.,

$$m \leqq m_i \leqq M_i \leqq M.$$

Hence

$$\Sigma m\overline{\overline{\delta}}_i \leqq \Sigma m_i\overline{\overline{\delta}}_i \leqq \Sigma M_i\overline{\overline{\delta}}_i \leqq \Sigma M\overline{\overline{\delta}}_i.$$

Thus

$$m\Sigma\overline{\overline{\delta}}_i \leqq S^{|}_D \leqq \overline{S}_D \leqq M\Sigma\overline{\overline{\delta}}_i. \tag{8}$$

But by § 376, 2,

$$\Sigma\overline{\overline{\delta}}_i = \overline{\overline{\mathfrak{A}}}. \tag{9}$$

So far as in my book. To get (7) put (9) in (8). The only point in this simple demonstration which Professor Fréchet can attack is the relation (9). But this brings the question around again to the fundamental relation (5) which we have already discussed. Professor Fréchet is thus repeating himself.

In connection with the relation (7) I quote the following remark of Professor Fréchet. He says: "Curiously enough Professor Pierpont did not think it useful to mention that the inequality (2) should hold." I suppose I did not mention the relation (2) because I thought it too self-evident. The proof is immediate:

Let $D_1, D_2 \cdots$ be an extremal sequence as defined on page 374. Then replacing D by D_n in (8) above we have $S_{D_n} \leqq \overline{S}_{D_n}$. Let now $n \doteq \infty$, we get the relation (2) by § 383, 1.

5. Finally at the close of Professor Fréchet's article I am told by him that I do not know the real significance of my own work. "The real difference is not as Professor Pierpont asserts for his own (definition) that it makes use of an infinite instead of a finite number of parts of E (as in Riemann's definition). It lies essentially, etc. . . ."

Professor Fréchet will pardon me if I still hold to my original opinion in spite of his illuminating remarks. He has been so often wrong, as I hope I have made clear, that he may well be wrong here also.

The rest of Professor Fréchet's remarks relate to matters of taste and as de gustibus non est disputandum I refrain from entering the controversy. Professor Fréchet claims that I have erred on three counts, viz.: 1. The relation (2). 2. The relation (5). 3. The relation (7).

The only proof he has adduced is an example whose validity depends on establishing the *vital* fact that $\mathfrak{A} = (\delta_1, \delta_2)$ is a separated division of $\mathfrak{A}$. I expect he will hasten to remove this lacuna.

JAMES PIERPONT.

YALE UNIVERSITY,
January, 1916.

CARMICHAEL'S MONOGRAPHS ON BRANCHES OF THE THEORY OF NUMBERS.

The Theory of Numbers. By R. D. CARMICHAEL. New York, John Wiley and Sons, 1914. 8vo. 94 pages. Price $1.

Diophantine Analysis. By R. D. CARMICHAEL. New York, John Wiley and Sons, 1915. 8vo. 6+118 pages. Price $1.25.

THE various series of tracts or monographs on mathematics which are in course of publication in several European countries are so well known and the arguments in favor of them are so generally conceded that it is not surprising that several series of tracts have been recently begun in America. In view of the purpose of such a tract, the editor of a series naturally imposes a definite upper limit to its length. Frequently the tract relates to a very extensive field of mathematics and the problem of the selection of topics presents a serious difficulty to the author.

There is an added difficulty in the case of a tract for beginners in the theory of numbers (and the same point would apply to the case of the theory of groups): the subject is somewhat abstract and the nature of the theorems and proofs is quite different from that to which the reader is accustomed. Consequently the author of the tract on the Theory of Numbers has wisely adopted a very elementary and expansive style of presentation, even at the expense of a reduction of the number of topics treated. A like motive, combined with the desire to emphasize methods rather than results, doubtless led the author to give several proofs of Fermat's theorem and Euler's generalization, although the space used could have been utilized for the presentation of further results.

Chapter I deals (in 23 pages) with the uniqueness of factorization into primes, the greatest common divisor and least common multiple of two or more integers, the highest power of a prime which divides $n!$, and the simplest properties of prime numbers.

Chapter II devotes 7 pages to Euler's ϕ-function or indicator. Two methods of evaluating $\phi(m)$ are given in detail, and a third method is suggested.

Chapter III gives in 10 pages the formal properties of congruences, a proof that a congruence of degree n with respect

to a prime modulus has at most n (real) roots, and the simpler theorems on linear congruences.

Chapter IV treats (in 14 pages) of Fermat's theorem and its extensions and converse, its application to linear congruences and to Euler's criterion for quadratic residues; also Wilson's theorem and its converse.

Chapter V devotes 15 pages to a rather full account of the theory of primitive roots. Concerning the important function $\lambda(m)$, the maximum indicator of Cauchy, it is proved that there exist integers belonging to the exponent $\lambda(m)$ modulo m. Thus $x^{\lambda(m)} \equiv 1 \pmod{m}$ is satisfied by every integer x prime to m, while this is not true for an exponent less than $\lambda(m)$.

Chapter VI gives (in 17 pages) a brief first view of various additional topics such as the theory of quadratic residues, including a statement of the law of reciprocity; Galois imaginaries, from the intuitive point of view of Galois; rational right triangles.

In view of its great clearness and elementary character, this book will prove a boon to the general reader desirous of an introduction to the "queen of the sciences," as well as to those students of mathematics who wish to acquire quickly and easily a working knowledge of the theory of numbers sufficient for its ordinary applications in other fields of mathematics. For the latter purpose, the text should have contained numerous exercises involving quadratic residues. On the other topics treated, the exercises are numerous and well selected. Having examined the full reports from a beginner in this subject who used this text for private reading, the reviewer is confident that the text is well suited for the two classes of readers mentioned above. In view of the limited range of topics, the book would have to be supplemented by lectures if adopted for a major course in the theory of numbers, as usually presented at the universities.

The author has deviated from custom in his definition of three or more relatively prime numbers. This term is used (page 9) when the numbers have no common factor except unity. According to Dirichlet,* numbers are relatively prime only when every pair of them are relatively prime. While Carmichael is consistent in the use of his definition, a student who was reading the text occasionally made errors by falling back upon the older (and perhaps more natural) definition.

In exercise 7, page 17, the word two should be inserted before

* Vorlesungen über Zahlentheorie, ed. 4, 1894, p. 11.

"relatively prime factors." There is an evident misprint in exercise 3, page 20.

The text on Diophantine analysis is of much greater scientific importance than the book just reviewed, since there exists no other single book in any language which presents so much of the material on Diophantine equations, and certainly no earlier book which undertakes such a systematic presentation of important aspects of the theory. From the time of Pythagoras there has been an uninterrupted interest in the subject now known as Diophantine equations. In particular, there has been for three centuries a special interest in Fermat's last theorem, that perpetual challenge to mathematical combat, that impenetrable armor upon which has been shattered the lance of many a gallant trained soldier, that lure which has fascinated only to repel the uncouth advances of many a camp follower and raw recruit. If only to satisfy a reasonable curiosity, the general mathematical public is entitled to a clear exposition of the problems, methods and results achieved in an ancient subject in which the theorems appear to be so simple and yet are often so difficult to prove.

Chapter I deals with the general nature of Diophantine analysis, the lack of general methods of investigation, rational oblique and right triangles, and Fermat's method of infinite descent.

Chapter II is an introduction to the application in this subject of a principle, discussed later in this review, which is really only the theorem that, in a given algebraic domain, the norm of a product equals the product of the norms of the factors. The applications here are to Pell's equation $x^2 - Dy^2 = \sigma^2$ (especially the case $\sigma = 1$), to $x^2 + ay^2 + bu^2 + abv^2 = t^2$, to the complete solution of $x^2 + y^2 + z^2 = t^2$ in integers, to the derivation of a second solution of $x^4 + ay^4 + bz^4 = t^2$ from a given solution, and to a like result for $x^4 + ay^4 = \mu^2 + b\nu^2$.

Chapters III and IV are devoted to Diophantine equations of the third and fourth degrees in two or more variables.

Chapter V devotes 19 pages to Fermat's last theorem: If n is an integer greater than 2, there are no integers x, y, z all different from zero, such that $x^n + y^n = z^n$. The formulas given by Abel and Legendre are proved. There is a presentation of the method of Sophie Germain as developed by

Legendre and recent writers. Four pages are used to give a summary of further results known about Fermat's last theorem, many of the results being stated in the form of exercises.

Chapter VI directs attention to the possibility of using rational solutions of functional equations as a means of classifying isolated problems on Diophantine equations. The illustration employed is the functional equation

$$(a^2 + 1)(u_a^2 + 1) = v_a^2 + 1.$$

Various solutions of this are employed in a treatment of Fermat's problem to find three squares such that the product of any two of which, added to the sum of those two, gives a square.

The chief aim of the author is set forth in the preface as follows. "The task of the author has been to systematize, as far as possible, a large number of isolated investigations and to organize the fragmentary results into a connected body of doctrine. The principal single organizing idea here used and not previously developed systematically in the literature is that connected with the notion of a multiplicative domain introduced in Chapter II" (that of applying the multiplicative property of norms of algebraic numbers). Again on page 50, the author says "The method of extending this set (of numbers $x_1^n + a_1x_1^{n-1}x_2 + \cdots + a_nx_2^n$) so that the resulting set shall form a domain closed with respect to multiplication grows out of a remark due to Lagrange (Oeuvres, 7, pages 164–179), though Lagrange seems nowhere to have utilized it in connection with Diophantine problems. A partial use of it has been made by Legendre (Théorie des Nombres, volume 2, ed. 3, pages 134–141); but its consequences seem nowhere to have been systematically developed."

This programme has been carried out so admirably that it does not detract from the value of the work to point out that Lagrange did apply the idea to Diophantine equations and that later writers developed the theory quite systematically.

Lagrange* proved that, if a is a fixed nth root of unity, the product of two functions of the type

$$p = t + ua\sqrt[n]{A} + xa^2\sqrt[n]{A^2} + \cdots + za^{n-1}\sqrt[n]{A^{n-1}}$$

is of like form. Hence if we replace a by the different nth

* *Mém. Ac. R. Sc. Berlin*, vol. 23, 1769; Oeuvres 2, 527.

roots of unity and form the product of the functions so obtained from p, we obtain a rational function P of $t, u, \cdots, z, A$, such that the product of two functions of type P is a third function of type P. It is shown how P can be found by elimination. The theory is applied to the solution of

$$(1) \qquad r^n - As^n = q^m.$$

We desire to express each factor $r - asA^{1/n}$ as an mth power p^m, where $a^m = 1$, and p is the above linear function. Then

$$p^m = T + Ua\sqrt[n]{A} + Xa^2\sqrt[n]{A^2} + \cdots + Za^{n-1}\sqrt[n]{A^{n-1}}.$$

Hence we take $r = T$, $s = -U$, $X = 0, \cdots, Z = 0$. Thus (1) is solvable by this method if $X = 0, \cdots, Z = 0$ are solvable. Although we have only $n - 2$ equations in n variables, they do not always have rational solutions. For the case $n = 3$, $m = 2$, the single condition $X = 0$ gives $x = -u^2/2t$; then

$$r = T = t^2 + 2Aux = t^2 - \frac{Au^3}{t},$$

$$-s = U = Ax^2 + 2tu = \frac{Au^4}{4t^2} + 2tu,$$

$$q = P = t^3 + Au^3 - 3Atux + A^2x^3.$$

For $n = m = 3$, the condition is $tu^2 + t^2v = Auv^2$; but Lagrange did not complete the discussion of this case.

The method just applied to the two cases having $n = 3$ was later extended by Lagrange* from the special case $a^3 = 1$ to the case in which a is a root of any cubic equation. This work is reproduced by Carmichael on pages 55, 56, where a reference to Lagrange would have been in place. For, although this reference was given five pages earlier, it was there stated (see quotation above) that Lagrange had not applied the idea to Diophantine problems. Carmichael reproduced Lagrange's use of the idea to obtain a set of solutions, involving two parameters, of

$$(2) \qquad x^3 + ax^2y + bxy^2 + cy^3 = v^2.$$

Lagrange remarked that his solution of (2) " is well worthy of notice on account of its generality and the manner in which

* Addition IX to Euler's Algebra, vol. 2, 1774, pp. 644–9; Oeuvres de Lagrange, vol. 7, pp. 170–9.

it was derived, which is perhaps the only way which can lead to it easily."

Carmichael attempts to apply the idea to the similar equation in which v^2 is replaced by v^3, but finds (page 58) that the condition $X = 0$ is so complicated that a complete solution is hardly to be expected; he then gives the recent methods by Schaewen, based on other principles.

Lagrange* made much use of the property

$$(p^2 - Bq^2)(p_1^2 - Bq_1^2) = (pp_1 \pm Bqq_1)^2 - B(pq_1 \pm qp_1)^2$$

in his various investigations on Diophantine equations of the second degree, especially in his work on Pell's equation and in the solution of $u^2 - Bt^2 = A$ in rational numbers. The corresponding formula (page 525) concerning

$$F = p^2 - Bq^2 - Cr^2 + BCs^2$$

was used by him in the proof that every number is expressible as a sum of four squares. G. Libri† used this property of F and gave a formula stated to give all the ways of reducing a product FF_1 of two such functions to a like form F_2; he gave (page 292) an identity expressing the product of two sums of four cubes as a sum of cubes of four rational expressions, and remarked that also

$$3x^4 + y^4 - z^4 - 3u^4$$

repeats under multiplication and represents rationally all rational numbers.

Lagrange‡ stated that "the simplest and most general method for equations like $x^4 + ay^4 = z^2$ is perhaps that by factors in his additions (final chapter) to Euler's algebra."

It is therefore clear that Lagrange was fully aware of the applications of the multiplicative property of norms to Diophantine problems. Moreover, it is rather evident that he developed this property of norms for the purpose of applying it to Diophantine equations.

As to Carmichael's remark that the consequences of this idea of Lagrange's "seem nowhere to have been systematically developed," it should be noted that the series of papers by

---

* Oeuvres, 2, p. 386, p. 523.

†*Journal für Math.*, vol. 9, 1832, p. 287.

‡ Oeuvres, 4, p. 395.

Desboves* give such a systematic development, perhaps as extensive as Carmichael's. Among the equations treated by this method by Desboves are

$$X^3 + rY^3 = Z^2,\ Z^3,\ Z^4; \quad \xi^4 + k\eta^4 = \zeta^2.$$

In this connection should be cited a general theorem of importance due to Dirichlet:† If at least one of the roots $\alpha, \beta, \cdots, \omega$ of an equation $s^n + as^{n-1} + \cdots + h = 0$ is real and if $a, \cdots, h$ are integers, while $s^n + \cdots$ has no rational divisor, and if

$$\phi(\alpha) = x + \alpha y + \cdots + \alpha^{n-1}z,$$

then the Diophantine equation

$$F(x, y, \cdots, z) \equiv \phi(\alpha)\phi(\beta) \cdots \phi(\omega) = 1$$

has an infinity of integral solutions. If the corresponding Lagrangian function P can assume a given value N, it takes the same value N for an infinitude of sets of values of $x, y, \cdots, z$. Poincaré‡ noted that, under the same conditions, the solution of $F = N$ reduces to the problem of forming all complex ideals of norm N, and discussed the latter question.

Carmichael (pages 44–48) has applied Lagrange's method to two new types of Diophantine equations

$$x^4 + ay^4 + bz^4 = t^2, \quad x^4 + ay^4 = \mu^2 + b\nu^2,$$

obtaining a second set of solutions from a given set. When applied to another equation (page 62), the method led to a solution which "unfortunately lacks generality," so that a special method was devised. The author himself of course recognizes that Lagrange's method is not a universal panacea.

If the strict limitations of space had not made it necessary for the author to develop a considerable body of classic results from a single central point of view, he would doubtless have given an exposition of other methods of considerable generality. For example, the elegant method in the joint paper§ by Hilbert and Hurwitz to obtain all sets of rational solutions of

* *Nouv. Ann. Math.*, ser. 2, vol. 18, 1879, pp. 265–279, 398–410, 433–444, 481–499.

† *Comptes Rendus*, Paris, vol. 10, 1840, pp. 285–8.

‡ Ibid., vol. 92, 1881, p. 777.

§ *Acta Mathematica*, vol. 14, 1890–1, pp. 217–224.

$f(x, y, z)=0$, where f is a homogeneous polynomial of degree n with integral coefficients such that the curve $f = 0$ is of genus (or deficiency or Geschlecht) zero. Again, Poincaré,* with the aim to find a bond between various problems of Diophantine analysis, has treated homogeneous polynomials $f(x, y, z)$ with integral coefficients from the standpoint of classes of curves $f = 0$ under birational transformations with rational coefficients. Finally, C. Runge† and E. Maillet‡ have given conditions for an infinitude of sets of solutions of any Diophantine equation in two variables. References to papers of this type would have been in place in such a brief text.

The reference (page 68) to Euler alone is rather generous, as Euler, in his proof of the impossibility of integral solutions, all different from zero, of $x^3 + y^3 = z^3$, did not give a rigorous proof of the vital point that, if $p^2 + 3q^2$ is a cube, it is the cube of a number $t^2 + 3u^2$ and that $p + q\sqrt{-3}$ is the cube of $t + u\sqrt{-3}$. For a proof, see Pepin.§

The numerous exercises are of three types with distinguishing marks. There are 133 exercises intended to develop facility in the handling of the subject; 53 additional exercises are of more difficulty and are intended primarily as a summary of known results not otherwise included in the text; while 35 further exercises are intended to suggest investigations. With many of the exercises are affixed names of writers and dates, but no journal references. The author evidently made a thorough examination of the extensive literature of the subject.

In view of its undoubted scientific merits, the book should be very useful to the student of Diophantine analysis. In view of its clear and attractive style of presentation, its emphasis on important results and methods, and its proper subordination of minor or more technical matters, the text should appeal strongly to the general reader desirous of obtaining in a brief time a clear view of this attractive branch of the theory of numbers.

L. E. DICKSON.

---

* *Jour. de Math.*, ser. 5, vol. 7 (1901), 161–233.
† *Jour. für Math.*, vol. 100 (1887), pp. 425–435.
‡ *Jour. de Math.*, ser. 5, vol. 6 (1900), pp. 261–277.
§ *Jour. de Math.*, ser. 3, vol. 1 (1875), p. 317.

NOTES.

THE opening (January) number of volume 38 of the *American Journal of Mathematics* contains the following papers: "The oscillation of functions of an orthogonal set," by O. D. KELLOGG; "On some properties of the medians of closed continuous curves formed by analytic arcs," by ARNOLD EMCH; "Theorems on the groups of isomorphisms of certain groups," by L. C. MATHEWSON; "Self-projective rational sextics," by R. M. WINGER; "On linear difference and differential equations," by C. E. LOVE; "The uniform motion of a sphere through a viscous liquid," by R. W. BURGESS; "Note on the theory of optical images," by GEORGE STEIĆ.

AT the meeting of the London mathematical society on January 13 the following papers were read: "The transition from vapor to liquid when the range of the molecular attraction is sensible," by J. LARMOR; "A note on the uniform convergence of the Fourier series $\Sigma a_n \sin n\theta$" and "A condition for the validity of Taylor's expansion," by T. W. CHAUNDY.

AT the meeting of the Edinburgh mathematical society on January 14 the following papers were read: "On the continued fractions of Tchebychef and Laguerre," by H. DATTA; "The conformal representation of the quotient of two Bessel functions," by A. MILNE.

THE following Cambridge tracts in mathematics and mathematical physics are announced as in press, to appear in a few weeks: The Definite Integral, its Meaning and Fundamental Properties, by E. W. HOBSON; An Introduction to the Theory of Attractions, by T. J. I'A. BROMWICH; Pascal's Hexagon, by H. W. RICHMOND; Lemniscate Functions, by G. B. MATHEWS; Chapters on Algebraic Geometry, by F. H. BAKER; The Integrals of Algebraic Functions, by F. H. BAKER.

DARTMOUTH COLLEGE. The following courses in mathematics will be given in the summer session, July 16 to August 16: By Professor J. W. YOUNG: The reorganization of secondary school mathematics.—By Professor E. G. BILL: Plane analytical geometry; Projective geometry.

THE University of Messina has been reopened with the following mathematical staff: V. MARTINETTI, professor of analytic geometry; Z. GIAMBELLI, formerly of the University of Cagliari, associate professor of projective and descriptive geometry; P. CALAPSO, of the University of Palermo, associate professor of algebra and analysis; E. LAURA, of the University of Turin, associate professor of mechanics and mathematical physics.

THE royal prize of 10,000 francs, conferred every six years by the Reale Accademia dei Lincei, has just been awarded to Professor FRANCESCO SEVERI, of the University of Padua, for his researches in algebraic geometry.

PROFESSOR G. H. BRYAN, of Bangor College, has been elected to an honorary fellowship in Peterhouse College, Cambridge.

PROFESSOR P. KOEBE, of the University of Jena, has been elected a corresponding member of the Göttingen academy.

PROFESSOR I. SCHUR, of the University of Bonn, has been appointed associate professor of mathematics at the University of Berlin, as successor to the late Professor KNOBLAUCH.

DR. A. FUNK has been appointed docent in mathematics at the German University of Prague.

DR. R. GRAMMEL has been appointed docent in mechanics at the technical school of Dantzig.

MR. C. GARLOUGH has been appointed instructor in mathematics at Wheaton College.

MISS S. F. RICHARDSON, assistant professor of mathematics in Vassar College, died February 2, 1916. Miss Richardson was a graduate of Vassar, and a member of the teaching staff since 1886. She became a member of the American Mathematical Society in 1905.

PROFESSOR J. W. R. DEDEKIND, of the technical school of Brunswick, died February 12, at the age of 83 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AL-KHOWARIZMI. Robert of Chester's Latin translation of the Algebra of Al-Khowarizmi. With an introduction, critical notes and an English version by L. C. Karpinski. (University of Michigan Studies. Humanistic series, vol. 11, part 1.) New York, Macmillan, 1915. Royal 8vo. 4+164 pp.+4 plates. Paper. $2.00

BERZOLARI (L.). Geometria analitica. II: Curve e superficie del secondo ordine. (Manuali Hoepli.) Milano, Hoepli, 1916. 11+427 pp. L. 3.00

BLICHFELDT (H. F.). See MILLER (G. A.).

DICKSON (L. E.). See MILLER (G. A.).

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band II 3, Heft 2: C. Runge und F. A. Willers, Numerische und graphische Quadratur und Integration gewöhnlicher und partieller Differentialgleichungen. Leipzig, Teubner, 1915. Gr. 8vo. Pp. 47–176. M. 4.20

FERROLLI (T.). Geometria non-euclidea. (Biblioteca del popolo, no. 590.) Milano, casa ed. Sonzogno (Matarelli), 1915. 16mo. 61 pp. L. 0.20

FONTSERÉ Y RIBA (E.). See TALLADA (F.).

GARBE (E.). Zur Theorie der Integralgleichungen. Tübingen, 1914. 8vo. 43 pp.

GIGER (A.). Ueber die dritte Steinersche Erzeugungsweise der Fläche dritter Ordnung. Zürich, 1914. 8vo. 30 pp.

JANSSEN (G.). Ueber die definitionsmässige Einführung der affinen und der adäquaten Geometrie auf Grund der Verknüpfungsaxiome. Göttingen, 1913. 8vo. 48 pp.

KARPINSKI (L. C.). See AL-KHOWARIZMI.

KLEIN (L.). Streifzüge in das Gebiet der Mathematik und Geometrie. Heft 1: Zur Kreislehre (über Näherungskonstruktionen für algebraisch unlösbare Aufgaben aus der Kreislehre); über das sogen. Vivianische Fenster. Heft 2: Ueber die Verallgemeinerung des Feuerbachschen Kreises. Korneuburg, 1915. Gr. 8vo. 43+32 pp. M. 1.00+1.00

KODWEISS (W.). Theorie der Monge-Ampèreschen Differentialgleichung mit drei unabhängigen Variablen. Tübingen, 1913. 8vo. 48 pp.

MAZKEWITSCH (D.). Ueber projektivische Strahlen- und Punktinvolutionen und einige Erzeugnisse derselben. (Thèse, Berne.) Zürich, Müller, Werder et Cie., 1915. 8vo. 76 pp.

MEINONG (A.). Ueber Möglichkeit und Wahrscheinlichkeit. Beiträge zur Gegenstands- und Erkenntnistheorie. Leipzig, Teubner, 1915. Gr. 8vo. 16+760 pp. M. 19.00

MILLER (G. A.), BLICHFELDT (H. F.) and DICKSON (L. E.). Theory and application of finite groups. New York, Wiley, 1915. 8vo. 390 pp. $3.00

Pascal (E.). I miei integrafi per equazioni differenziali. Napoli, Pellerano, 1915. 16mo. 137 pp. L. 6.00

Pflüger (G.). Die Formschönheit einfacher geometrischer Gebilde. Bausteine zu einer wissenschaftlichen Aesthetik. Stuttgart, 1915. Gr. 8vo. 47 pp. M. 2.50

Plaut (H. C.). Ueber gemeinsame Teilbarkeit von n Formen einer Variabeln von n linearen homogenen Differential- oder Differenzenausdrücken. Königsberg, 1914. 8vo. 73 pp.

Robert of Chester. See Al-Khowarizmi.

Roth (L.). Ueber die singulären Stellen des Haupttangentenkurvensystems einer Fläche. München, 1914. 8vo. 50 pp.

Runge (C.). See Encyklopädie.

Straszewicz (S.). Beiträge zur Theorie der konvexen Punktmengen. Zürich, 1914. 8vo. 57 pp.

Tallada (F.). Consideraciones acerca del espacio. Con discurso de contestacion por E. Fontseré y Riba. Barcelona (*Mem. Acad.*), 1914. 4to. 20 pp. M. 2.00

Timerding (H. E.). Die Analyse des Zufalls. Braunschweig, 1915. 8vo. 9+167 pp. M. 5.00

Volterra (V.). The theory of permutable functions. Princeton, N. J., 1915. 8vo. 68 pp. $1.25

Walter (J. E.). Nature and cognition of space and time. West Newton, Pa., Johnston and Penney, 1915. 186 pp.

Watson (G. N.). See Whittaker (E. T.).

Weikersheimer (S.). Studien zur Integration homogener linearer Differentialgleichungen durch bestimmte Integrale. Würzburg, 1914. 8vo. 40 pp.

Whittaker (E. T.) and Watson (G. N.). A course of modern analysis. 2d edition. Cambridge, University Press, 1915. 560 pp. 18s.

Willers (F. A.). See Encyklopädie.

II. ELEMENTARY MATHEMATICS.

Dobbs (F. W.) and Marsden (H. K.). Arithmetic. Part 1. (Bell's mathematical series for schools and colleges.) London, Bell, 1915. 8vo. 18+353+24 pp. 3s.

F. G. M. Manuel d'algèbre d'après les programmes de 1902 et de 1912. (Cours de mathématiques élémentaires.) Tours, A. Mame; et Paris, J. de Gigord, 1915. 8vo. 16+562 pp.

Marsden (H. K.). See Dobbs (F. W.).

Merriman (M.). Mathematical tables for classroom use. New York, Wiley, 1915. 8vo. 68 pp. $0.50

III. APPLIED MATHEMATICS.

Arndt (K.). Handbuch der physikalisch-chemischen Technik für Forscher und Techniker. Stuttgart, 1915. Gr. 8vo. 16+830 pp. M. 28.00

Auerbach (F.). See Handbuch.

BALLIF (L.). See SENSEVER (G.).

BAULE (B.). Theoretische Behandlung der Erscheinungen in verdünnten Gasen. Göttingen, 1914. 8vo. 34 pp. M. 1.50

BLANCHARD (A. H.). Elements of highway engineering. New York, Wiley, 1915. 8vo. 526 pp. $3.00

BÖTTGER (H.). Physik. Zum Gebrauch bei physikalischen Vorlesungen in höheren Lehranstalten sowie zum Selbstunterricht. Band 2: Optik, Elektrizität, Magnetismus. Braunschweig, 1915. Gr. 8vo. Pp. 12+895–2117. Leinenband. M. 26.00

BROCKMANN (K.). Die Bewegung der Ionen im Glimmstrom. Berlin, 1915. 8vo. 34 pp. M. 2.00

BRYANT (W. W.). A history of astronomy. 7s. 6d.

CALDERWOOD (J. P.). See MOYER (J. A.).

CICCONETTI (G.). La latitudine astronomica dell'osservatorio vesuviano determinata nel 1910 (r. Commissione geodetica italiana). Bologna, tip. Gamberini e Parmeggiani, 1915. 4to. 13 pp.

CUNNINGHAM (E.). Relativity and the electron theory. New York, Longmans, 1915. 8vo. 8+96 pp. $1.10

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band V 3, Heft 3: M. v. Laue, Wellenoptik; mit Beitrag über spezielle Beugungsprobleme von P. S. Epstein. Leipzig, Teubner, 1915. Gr. 8vo. Pp. 359–525. M. 5.20

EPSTEIN (P. S.). See ENCYKLOPÄDIE.

GEELMUYDEN (H.). Laerebog i astronomi. 2. udgave. Christiania, 1915. 8vo. 278 pp. M. 12.00

GRÄTZ (L.). See HANDBUCH.

HAMMER (W.). Ueber eine direkte Messung der Geschwindigkeit von Wasserstoffkanalstrahlen und über die Verwendung derselben zur Bestimmung ihrer spezifischen Ladung. Freiburg, 1913. 8vo. 49 pp.

HANDBUCH der Elektrizität und Magnetismus. Bearbeitet von F. Auerbach, W. Jäger, E. Riecke u. a. herausgegeben von L. Grätz. (5 Bände.) Band 3, Lieferung 2 und Band 4, Lieferung 2. Leipzig, Barth, 1915. Gr. 8vo. Pp. 5+181–350+6+271–710. M. 22.80

HESS (H. D.). Graphics and structural design. New York, Wiley, 1915. 8vo. 444 pp. $3.00

HOUSDEN (C. E.). Is Venus inhabited? New York, Longmans, 1915. 39 pp. Cr. 8vo. Paper. $0.50

IVES (H. C.). See SEARLES (W. H.).

JÄGER (W.). See HANDBUCH.

KENT (W.). Mechanical engineers pocket book. 9th edition, revised. New York, Wiley, 1915. 12mo. 1567 pp. Leather. $5.00

LAUE (M. v.). See ENCYKLOPÄDIE.

LAWS (B. C.). Stability and equilibrium of floating bodies. 10s. 6d.

LECHNER (G.). Untersuchungen der Turbulenz beim Durchströmen von Wasser und Quecksilber durch spiralförmig gewundene Kapillaren. Würzburg, 1913. 8vo. 35 pp.

MACH (E.). Kultur und Mechanik. Stuttgart, 1915, 8vo. M. 3.00

MARCOLONGO (R.). Il problema dei tre corpi da Newton (1686) ai nostri giorni. Pisa, stab. tip. Toscano, 1915. 8vo. 102 pp.

MIRINNY (L.). Pantosynthèse. Fonction pandynamique. Etude primordiale abrégée avec une planche hors texte et un portrait de l'auteur. Paris, Imprimerie Normande, 1915. 12mo. 50 pp.

MORET (J.). L'emploi des mathématiques en économie politique. Paris, Giard et E. Brière, 1915. 8vo. 272 pp. Fr. 6.00

MOYER (J. A.) and CALDERWOOD (J. P.). Engineering thermodynamics. New York, Wiley, 1915. 8vo. 203 pp. $2.00

OELSNER (A.). Ueber innere Reibung bei tautoren Flüssigkeiten. Breslau, 1913. 8vo. 39 pp.

RIECKE (E.). See HANDBUCH.

RUTHS (C.). Neue Relationen im Sonnensystem und Universum. Darmstadt, 1915. Gr. 8vo. 10+162 pp. M. 4.00

SALINGER (H.). Ueber die Aequipotentialflächen in der positiven Lichtsäule des Glimmstromes. Berlin, 1915. 8vo. 31 pp. M. 1.50

SCHÖLER (K.). Ueber das Verhältnis $k = C_p/C_v$ der spezifischen Wärmen von Gasen bei konstantem Druck und bei konstanten Volumen bei verschiedenen Drücken. Kiel, 1914. 8vo. 35 pp. M. 1.50

SCHULTZ (J.). Ueber eine von J. L. Lagrange gegebene trigonometrische Interpolationsmethode und deren Anwendung auf Kosmophysik. München, 1913. 8vo. 68 pp.

SEARLES (W. H.) and IVES (H. C.). Field engineering. 17th edition, revised and reset. New York, Wiley, 1915. 12mo. 23+632 pp. Leather. $3.00

SEIDEL (R.). Ueber starre räumliche Bewegungen, deren Achsenflächen Zylinder sind. Dresden, 1914. 8vo. 143 pp.

SENSEVER (G.) et BALLIF (L.). Le combat aérien. (Etude cinématique.) Paris, Librairie aéronautique, 1915. 8vo. 172 pp. Fr. 3.00

TALBOT (F. A.). Submarines: their mechanism and operation. London, Heinemann, 1915. 10+274 pp. 3s. 6d.

TREFFTZ (E.). Ueber die Kontraktion kreisförmiger Flüssigkeitsstrahlen. Strassburg, 1914. 8vo. 56 pp.

TURNER (H. H.). A voyage in space. London, Society for the Propagation of Christian Knowledge. 16+304 pp. 6s.

WEIDNER (E. F.). Handbuch der Babylonischen Astronomie. 1ter Band: Der Babylonische Fixsternhimmel. Beiträge zur ältesten Geschichte der Sternbilder. 1te Lieferung. Leipzig, 1915. Lex. 8vo. 4+146 pp. M. 18.00

WERNER (H.). Messung von Wellenlängennormalen im internationalen System für den roten Spektralbereich. Tübingen, 1914. 8vo. 25 pp. M. 1.50

WILSON (W. L.). Elements of railroad track and construction. 2d edition, revised and enlarged. New York, Wiley, 1915. 8vo. 402 pp. $2.50

WOLFF (H.). Die Schwerkraft auf dem Meere und die Hypothese von Pratt. Berlin, 1913. 8vo. 117 pp. M. 2.00

SOME REMARKS ON THE HISTORICAL DEVELOPMENT AND THE FUTURE PROSPECTS OF THE DIFFERENTIAL GEOMETRY OF PLANE CURVES.*

BY PROFESSOR E. J. WILCZYNSKI.

PROBABLY the most fundamental characteristic of the human mind is its hatred for contradictions. All of our thinking is fundamentally influenced by this dislike; and the rôle of the mathematician, in his relation to reality, may be described in a fairly adequate manner by saying that it is his business to remove all contradictions from our discussions and, by gradually extending the scope of these discussions, to show that the world as a whole is thinkable.

To justify the validity, in the purely mathematical sense, of any construction of the intellect, absence of contradictions is necessary and sufficient. But the mere absence of contradictions from a realm of thought does not necessarily give it that essential artistic and harmonious one-ness which leads us to think of it as a unit. A peculiarity of the human mind, almost as important as its hatred for contradictions, is its dislike for sudden and frequent changes in the point of view. Thus, quite apart from the obvious practical difficulties of studying plants and stars and souls at the same time, the mind for the sake of its own peace and convenience, following its desire to move along a straight line, has divided knowledge into compartments, and refuses to think of more than one of these compartments at the same time. This procedure does not disturb in the least the profound conviction, present I believe in all thinkers, that at some future time from some other higher point of view the separateness of these compartments will be abolished. Indeed, we cannot help but think that a thoroughgoing unification of each separate realm is the best possible preparation for an ultimate and complete generalization which shall include the whole.

It is my purpose to-day to try and show you how one funda-

* Address of the retiring chairman of the Chicago Section of the American Mathematical Society, read at Columbus, Ohio, December 30, 1915.

mental idea has dominated and still dominates in the realm of differential geometry. I shall confine my discussion to the differential geometry of plane curves primarily on account of the comparative ease with which we can visualize configurations in the plane. But it is a notable fact that even in this limited domain so many problems have remained untouched, problems which it is easy to formulate and not difficult to solve. Some of the notions which I shall discuss, although conceived in admirable fashion nearly a century ago, have remained practically unnoticed. You would look for them in vain in any of our modern treatises, although these contain many other things of far greater difficulty and of much less interest. In fact, so completely have these ideas been neglected, that most mathematicians are probably under the impression that the differential geometry of plane curves is a very much restricted and uninteresting field not at all adapted for further research. Nevertheless it is true that scarcely one of the notions which will arise in our discussion to-day has been studied as fully as it deserves, while most of them have not as yet received any consideration whatever.

The notion of a general plane curve is at bottom identical with the general notion of function. It is obviously impossible to prove, by documentary evidence, that the ancient geometers did not possess the idea of a general curve. But we may assert, I think, that the available evidence indicates that the ancients knew how to deal only with very special curves such as straight lines, circles, conics, and a few other curves, including certain spirals. The relation between the notions "function" and "curve" only became evident in the seventeenth century of our era, when Descartes and Fermat had laid the foundations of analytic geometry, and it was the recognition of this relation which brought the notion of a general curve into the consciousness of mathematicians everywhere. But analytic geometry did more than merely formulate the notion of a general curve; it also provided a method for its investigation. If the first fruit of this union between analysis and geometry seemed to be of profit primarily for geometry by providing it with a new and limitless field for research, it soon became apparent that the union was to be profitable for analysis also. For the geometric problems which arose in this connection, such as the construction of a tangent to a curve with a given equation, the determination of the length of a

given arc of the curve, the calculation of areas bounded wholly or partly by curved lines, led inevitably toward the invention of the calculus. In fact, the method employed by Fermat, for instance, for determining the tangent of a curve really involved the essential processes of the differential calculus. This method is very simple and consists in formulating the definition of a tangent as follows. Take a point P on the curve, the point whose tangent we wish to draw. Join P to a second point Q which is also on the curve, and let us seek the limiting position which the line joining P to Q approaches, when Q, moving always along the curve, approaches P as a limit. This limiting position of the secant PQ is called the tangent.

Of course it was evident that the tangent would pass through P; the only problem was that of finding the direction of the tangent. The calculus, as developed by Newton and Leibniz, made it an easy matter to translate Fermat's definition of a tangent into the language of analysis and then actually to determine, by calculation, the direction of the tangent for a very extensive class of curves. Thus the problem of tangents could be regarded as solved. The normal could then be defined as a line perpendicular to the tangent at the point of contact, and its determination offered no further difficulty. It is worthy of remark, however, that one of the methods proposed by Descartes determines the normal first and the tangent afterward. We shall amplify this remark, a little later.

The analytic formulation of the notion of radius of curvature soon followed. Let us take three points P, Q, and R on a curve and let us pass a circle through these points. As Q and R move along the curve, approaching P as a limit, this circle will in general approach a limiting position which is called the circle of curvature. Its radius is called the radius of curvature, and its center the center of curvature. The familiar formula for the radius of curvature was published, apparently for the first time, in Newton's "Methodus Fluxionum" of 1736, although the notion itself was much older and was applied with great success by Huygens as we shall see immediately.

If the curve under consideration is itself a circle, the circle of curvature of every one of its points is of course that same circle. But if the original curve is not a circle, each of its points will have in general a different circle of curvature and the problem arises to find the locus of the centers of all of these

circles. This locus, called the evolute of the given curve, was first considered by Huygens, without of course making use of the notations of the calculus, in his "Horologium Oscillatorium" published in 1673. Huygens saw that the normals of the original curve would be the tangents of the evolute, and that the original curve could be regarded as the locus of the end-point of a string which was being gradually unwound from the evolute. This showed him that the difference between the lengths of two radii of curvature of the original curve was equal to the length of the corresponding arc of the evolute. Since the length of the string could be changed, it became apparent that, while every curve has only one evolute, it is itself the evolute of infinitely many other curves which are called its involutes. These involutes moreover are clearly the orthogonal trajectories of the tangents of the given curve.

For Huygens these notions were of great importance as applied to the special case of the cycloid, a curve first considered by Galileo. For he had recognized the isochronous property of the cycloid and had therefore shown that the cycloid was the most desirable curve for the oscillations of a pendulum. The problem now arose to devise a method which would compel the oscillations of a pendulum to take place along a cycloid. Huygens observed in this connection that a weight attached to a string would describe an arc of a cycloid if the string were attached in such a way as to cause it to wind and unwind along checks which had been given the form of an evolute of a cycloid. In fact it was this problem which led Huygens to his general theory of evolutes. He now found the remarkable theorem that the evolute of a cycloid is an equal cycloid, a theorem which must have appeared to him as a beautiful manifestation of the divine harmony of geometry.

The tangent may be regarded as a first approximation to the given curve in the neighborhood of one of its points. In modern terminology we may say that the tangent serves as a geometric image for the first derivative $f'(x)$ of the function $f(x)$, whose graph is the curve under consideration. The circle of curvature may be regarded as a second approximation to the curve or as a geometric image of the second derivative of $f(x)$. For a long time no attempt was made to find a geometric image for the third derivative. In 1841, however, Abel Transon published his beautiful "Recherches sur la courbure des lignes et des surfaces," in which he takes not only

this step but also the next, by devising appropriate geometric images both for the third and fourth order derivatives.

Transon first introduces the new notion which he calls "déviation" and which has been translated by the term "aberrancy" in the very few places where any notice has been taken of Transon's work. This notion is as follows. Let P be a point on a curve and let a chord be drawn parallel to the tangent at P and very close to this tangent. Let Q and R be the points (in the neighborhood of P) in which this chord intersects the curve, and let L be the middle point of QR. As QR approaches the tangent at P as a limit, the line PL will in general approach a limiting position called the axis of aberrancy. Let δ be the angle which the axis of aberrancy makes with the normal. The tangent of this angle is given by the formula

$$\text{(1)} \qquad \tan\delta = \frac{dy}{dx} - \frac{[1 + (dy/dx)^2]}{3(d^2y/dx^2)^2}\frac{d^3y}{dx^3}.$$

and Transon calls $\tan\delta$ the aberrancy of the curve at the point P. It would obviously be equal to zero at any point of a circle.

From familiar properties of conic sections it follows at once that the axis of aberrancy at any point of a conic is the line which joins this point to the center of the conic. Let us observe further that the expression (1) for the aberrancy contains only the first, second, and third derivatives of $y = f(x)$, so that the aberrancy is indeed adapted for the purpose of visualizing the third derivative.

The equation of a conic contains five essential constants, and a conic is therefore determined by five of its points. The condition that one curve shall have third order contact with another, may be expressed by saying that four of the points of intersection of the two curves coincide. Consequently there exists a one-parameter family (a pencil) of conics each of which has third order contact with the given curve at the given point P. All of these conics pass through P; they all have the same tangent, the same circle of curvature, and the same axis of aberrancy at P. Among these conics there will be one and only one parabola; let us call it the osculating parabola. Since, as we have just seen, the axis of aberrancy of the point P of our given curve is also the axis of aberrancy for each of the conics which has third order contact with our curve at P, and since the axis of aberrancy for a point of a conic passes

through the center of the conic, and since finally the center of a parabola is at infinity, we see that the axis of the osculating parabola is parallel to the axis of aberrancy.

To locate the axis of the osculating parabola completely we have, in addition to the remark just made, the following simple construction also due to Transon. Let D be the orthogonal projection of the center of curvature C upon the axis of aberrancy, and let E be the orthogonal projection of D upon the normal. Then the axis of the osculating parabola will pass through E. Since it must also be parallel to the axis of aberrancy we may now regard the axis of the osculating parabola as known.

The focus of this parabola must clearly be on a line through the point P which makes with the normal at P an angle equal to that made with this same normal by the axis of aberrancy. Thus the focus will be the intersection of this line with the axis. The directrix will of course be perpendicular to the axis. It will also pass through a point on the normal on the convex side of the curve whose distance from P is equal to half the radius of curvature. This last result is obtained by Transon as a corollary from a more general theorem. He considers the one-parameter family of parabolas, each of which has second order contact with the given curve at P. The directrices of all of these parabolas pass through the point just described, and the locus of their foci is a circle whose diameter is that half of the radius of curvature which terminates at P.

Among the conics which have third order contact with the given curve at P, there will be one for which the order of contact rises to the fourth order at least. This is the osculating conic of the point P and may be regarded as having five consecutive points in common with the curve. In common with all of the conics having third order contact with the given curve at P, it has its center on the axis of aberrancy, and the position of the center on the axis of aberrancy may therefore be regarded as a geometric equivalent for the value which the fourth derivative of $y = f(x)$ assumes at the point P. The properties which I have mentioned suffice to determine the osculating conic, but Transon develops some further theorems which facilitate its construction very considerably. I shall quote some of these on account of their great geometric interest.

These theorems are again concerned with the pencil of conics which have third order contact with the given curve at P. Transon finds that the axes of these conics envelop a parabola, whose directrix is the axis of aberrancy, and whose focus is the orthogonal projection of the center of curvature upon the line which joins P to the focus of the osculating parabola. This latter line is very easy to construct since, as we have seen, the normal bisects the angle between this line and the axis of aberrancy. The auxiliary parabola of Transon, I should like to add, also touches the lines which are tangent and normal to our original curve at P. As a consequence of this theorem of Transon's the directions of the principal axes of the osculating conic are obtained by drawing the two tangents from the center of the osculating conic to the auxiliary parabola.

The following property, not mentioned by Transon, is also of interest. Among the conics which have third order or four-pointic contact with the given curve at P, there will be infinitely many hyperbolas and ellipses, but in general no circle. This indicates the existence, in this family of conics, of a unique ellipse of minimum eccentricity. As the center of such a four-pointic conic moves along the axis of aberrancy on the concave side of the curve, starting from P, the eccentricity e of the corresponding ellipse will decrease continuously from unity toward the minimum value e_1, and will then increase, approaching the limit 1 as the center recedes beyond bound. Thus every value of e, between the minimum value e_1 and unity, will be attained twice. It is not difficult to show that the center O of the ellipse of minimum eccentricity and the centers, O_1 and O_2, of any two of these ellipses which have the same eccentricity are so related that PO is the geometric mean of OP_1 and OP_2. In general this four-pointic ellipse of minimum eccentricity is quite distinct from the osculating conic. It will coincide with the osculating conic however if and only if the original curve is a logarithmic spiral. Moreover, the eccentricity of the four-pointic ellipse of minimum eccentricity will in general change as P moves along the given curve. It will be constant if and only if the given curve is a logarithmic spiral.

We have seen how many interesting questions present themselves when we attempt to explore the relations between these various configurations all of which are determined by

the properties of a given curve in a single one of its points. But each of these configurations gives rise to new problems if we think of the point P as moving along the given curve. Thus, in the simplest case, already mentioned, the locus of the centers of curvature defines a new curve, which is at the same time the envelope of the normals which is called the evolute. In similar fashion we may investigate the envelope of the axes of aberrancy of a given curve. This envelope is at the same time the locus of the centers of its osculating conics. We may also study the envelopes of the axes and directrices of the osculating parabolas, the loci of their foci and vertices, and it is easy to formulate corresponding problems for the osculating conic and for the four-pointic ellipse of minimum eccentricity. Only a mere beginning of such a theory is available at present and most of this is due to Cesàro who made use of some of these notions in his "Lezioni di Geometria intrinseca," with the main emphasis however not upon the general theory but upon the application to certain well-known simple curves.

But I wish to call your attention primarily to a general method which enables us to push still farther the investigation of the properties of a curve in the vicinity of one of its points. The details which I have presented will serve to make clear the following general notions. Let us consider an equation of the form

$$\varphi(x, y; a_1, a_2, \cdots, a_n) = 0 \tag{2}$$

which involves the coordinates (x, y) of a point and n essential constants $a_1, a_2, \cdots, a_n$. For every set of values $a_1, a_2, \cdots, a_n$ this equation represents a curve. All of the curves represented by such an equation are said to form an n-parameter family. The condition that a curve of this family should pass through a given point gives rise to one relation between the parameters $a_1, \cdots, a_n$. Thus, in general, one curve of the family, or a finite number of such curves, is determined by the condition that the curve shall pass through n given points. If the parameters enter the equation (2) linearly, as is most frequently the case, there will be just one such curve passing through n given points, if we leave aside certain exceptional cases.

If then we take n points on any given curve, there will exist in general a unique curve of the n-parameter family (2)

which passes through them, and this curve will ordinarily approach a definite curve of the n-parameter family as a limit, if the n given points approach coincidence. The resulting limit curve will be said to *osculate* the given curve at the given point, in as much as it will have closer contact (n-pointic or $(n - 1)$th order contact) with the given curve at the given point than any other curve of the class defined by (2). To illustrate this notion we observe that the straight lines of the plane form a two-parameter family and that the tangent is the osculating straight line; that the circles form a three-parameter family and that the circle of curvature is the osculating circle. Moreover we have already made use of this terminology in our discussion of the osculating parabola and the osculating conic.

But our previous discussion shows quite clearly that, besides the osculating curves of a given class, those curves of the class are also of very great interest for which the order of contact falls short of the maximum by a single unit. Let us speak of such curves as *penosculants*. Evidently, from what has been said, the locus of the centers of the penosculating circles of a given point of a curve is the corresponding normal. In fact it was by means of this remark, to which we have already alluded very briefly, that Descartes defined the normal and thus indirectly solved the problem of tangents. Again we have seen that the locus of the foci of the penosculating parabolas is a circle whose diameter is equal to half the radius of curvature, and that the locus of the centers of the penosculating conics is the axis of aberrancy. Several others of the theorems which we have mentioned may be expressed more compactly by the help of this new terminology.

After this preliminary discussion, let us inquire what classes of osculants and penosculants ought to be introduced for the purpose of providing geometric interpretations for the derivatives of order higher than the fourth. For we may regard the theory outlined so far as exhaustive for the first four orders.

The general curve of the third order contains nine essential constants. Therefore the osculating cubic has nine consecutive points in common with the given curve at P. The penosculating cubics have eight consecutive points in common with the given curve at P and form a pencil. One of the cubics of this pencil is of special interest because it has a double point at P. Of course one of its double point tangents touches the

given curve at P; the other one crosses the curve at a non-vanishing angle. That branch of the penosculating nodal cubic which actually touches the given curve has seven-pointic or sixth order contact with it, and we may therefore use this cubic as a geometric image for the sixth derivative. The osculating cubic may of course be regarded as a representative of the eighth derivative.

The osculating cubic and the penosculating cubics were introduced by Halphen in his thesis on differential invariants in 1876. Halphen's interest was centered primarily upon the following feature of this situation. All of the penosculating cubics have, besides the eight points of intersection which are concentrated at P, a ninth point of intersection which I have called the Halphen point. Now it may happen that this ninth point also coincides with P, in which case P is called a coincidence point. This will happen if a certain projective differential invariant (in my notation the invariant θ_8) vanishes, and Halphen made use of these geometric notions for the purpose of calculating this invariant. He also showed that a curve may be composed entirely of coincidence points and that all such curves may be obtained from a certain logarithmic spiral by projective transformation.

We have now obtained representative osculants for contact of the first four orders and for orders six and eight. It remains to fill the gap for orders five and seven. In order to do this, we may consider special kinds of cubics distinguished by fundamental metric properties, just as the gap between the osculating straight line and the osculating conic was filled by means of osculating curves of order two distinguished by metric properties, namely by the osculating circle and parabola.

A parabola is a conic which touches the line at infinity. We may define a parabolic cubic to be a cubic which touches the line at infinity. Such a cubic contains eight arbitrary constants indicating the existence of osculating parabolic cubics to represent the seventh derivative. These cubics are not however necessarily the best adapted for this purpose. There are other cubics which accomplish as much and which are more easily accessible, as for instance that particular one of the penosculating cubics whose asymptote is parallel to the normal.

A circle may be defined as a conic which contains the so-called circular points at infinity. I shall not attempt to explain

this notion which is familiar to all mathematicians. A cubic which contains the circular points is called a circular cubic. The general equation of such a cubic contains seven arbitrary constants, indicating the existence of an osculating circular cubic which has seven-pointic or sixth order contact with the given curve, and which will probably for most purposes be a better representative of the sixth derivative than the one mentioned before.

The penosculating circular cubics of the point P all have six consecutive points in common with the given curve at P. Besides they intersect each other in the two circular points at infinity. The determination of their ninth point of intersection and the condition for its coincidence with P offer problems which are strictly analogous to the corresponding problems involving the Halphen point. Among the penosculating circular cubics there is one which has a double point at P. There is also one whose real asymptote is parallel to the normal. This latter curve may serve as an image for the derivative of the fifth order. But of course there are other cubics, both circular and non-circular, which may serve as well.

We may continue in this way to build up a theory of osculating algebraic curves of gradually increasing order, the osculants of each order being classified further according to the number of times that they pass through the circular points and the number of their asymptotes which are parallel to the normal. However I shall refrain from any further detailed exposition in this direction, only stopping to say that the analytic difficulties involved in actually determining the equations of these osculants up to and including the osculating cubic are far from insuperable. In fact, I actually have most of these equations at my disposal and they are much simpler than one might expect.

The penosculating nodal cubic is of fundamental importance in projective geometry. The following remarks may serve to make this clear. By introducing a suitably chosen system of homogeneous coordinates and denoting the properly chosen ratios of these homogeneous coordinates by X and Y, the equation of any curve, if it is neither a straight line nor a conic, may be expanded in the form

$$Y = \tfrac{1}{2}X^2 + X^5 + A_7X^7 + \cdots$$

in the vicinity of any one of its ordinary points. In this expansion A_7 and all of the remaining coefficients will be absolute projective invariants. The simplicity of this expansion, and the uniqueness of its form, make it evident that the system of coordinates, to which this expansion corresponds, must be of peculiar importance. But this system of coordinates is determined entirely by the properties of the penosculating nodal cubic. The triangle of reference has for two of its sides the double point tangents of this cubic and for its third side the line upon which lie its three points of inflection. Incidentally we may remark here that the relation of the simple quintic curve

$$Y = \tfrac{1}{2}X^2 + X^5$$

to the given curve is worthy of notice.

The fundamental importance of the penosculating nodal cubic also appears when we attempt to interpret the simplest of all projective integral invariants. This integral corresponds so nearly to the notion of length of arc, which is the invariant integral of lowest order in the metric theory, as to justify the prediction that it will be found to be of the greatest importance in future developments of the projective theory. Thus, for instance, this integral enables us to formulate at once a notion which generalizes, in the sense of projective geometry, Cesàro's intrinsic equation of a curve.

We have discussed osculating and penosculating curves of many different kinds, all of which however were algebraic. Transon also mentions the availability of the notion of an osculating logarithmic spiral, and in his projective theory Halphen makes use of the notion of an osculating anharmonic curve. The logarithmic spiral and the general anharmonic curve are transcendental curves to be sure, but they belong to a particularly simple type of transcendental curves. In fact, most of these curves are so closely related to algebraic curves that Leibniz thought it inadvisable to speak of them as transcendental, and invented a special name for them, calling them interscendental curves. So far as I am aware, no other curves, except those mentioned, have ever been used as osculants in the theory of plane curves.

Each of the osculants and penosculants which we have introduced has a function to perform which may be illuminated by an aphorism; the osculant is the microscope of the geometer.

Thus, to the naked eye the courses of two curves, which at a common point have the same tangent and the same radius of curvature, are in the vicinity of that point so nearly identical as to make them appear indistinguishable. The introduction of the notions of axis of aberrancy and osculating parabola serves to magnify the differences between the two curves in such a way as to enable us to distinguish between them. Again, if the two curves also have their osculating parabolas in common, we may judge of their divergence by means of their osculating conics. Thus the notion of osculant serves the differential geometer for the same purpose as does the microscope in the laboratory of the biologist. It magnifies the infinitesimal differences between two different curves sufficiently to cause them to make an emphatic impression upon the mind.

Thus the notions, osculant and penosculant, are the fundamental concepts of differential geometry. The systematic investigation of the magnitudes, loci and envelopes determined by the various classes of osculants and penosculants and the relations which exist between them makes up the whole subject matter of differential geometry. Differential properties of a general curve are merely integral properties of its osculants and penosculants.

THE UNIVERSITY OF CHICAGO,
December, 1915.

A CERTAIN SYSTEM OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS.

BY DR. H. BATEMAN.

(Read before the American Mathematical Society, February 26, 1916.)

1. It is known that if a function $V(x_1, y_1, z_1, t_1; x_2, y_2, z_2, t_2; \cdots; x_n, y_n, z_n, t_n)$ satisfies the system of $\frac{1}{2}n(n+1)$ partial differential equations*

$$(1)\quad \frac{\partial^2 V}{\partial x_p \partial x_q} + \frac{\partial^2 V}{\partial y_p \partial y_q} + \frac{\partial^2 V}{\partial z_p \partial z_q} = \frac{\partial^2 V}{\partial t_p \partial t_q} \quad (p, q = 1, 2, \cdots, n)$$

it becomes a solution of the reduced system of $\frac{1}{2}(n-1)n$

* See for instance H. Bateman, *Messenger of Mathematics*, March, 1914, p. 164.

equations* when the point (x_n, y_n, z_n, t_n) coincides with $(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1})$. Such a function V will be called *a multiple wave function*† *of rank* n and will be denoted by $V^{(n)}$ when we wish to indicate its rank.

It is easy to prove that such functions exist, for if we write

$$\begin{aligned} \alpha_p &= (x_p - iy_p)e^{i\omega} - i(z_p \pm t_p) \\ \beta_p &= (x_p + iy_p)e^{-i\omega} - i(z_p \mp t_p) \end{aligned} \tag{2}$$

the function

$$V = \int_0^{2\pi} f(\alpha_1, \alpha_2, \cdots, \alpha_n; \beta_1, \beta_2, \cdots, \beta_n; \omega)d\omega \tag{3}$$

satisfies the system of equations (1) and possesses the property just mentioned, f being an arbitrary function with finite second derivatives. Let us now consider two vector functions $H^{p,q}$ and $E^{p,q}$ whose components are defined by equations of type

$$H_x^{p,q} = \frac{\partial^2 V}{\partial y_p \partial z_q} - \frac{\partial^2 V}{\partial y_q \partial z_p}, \quad E_x^{p,q} = \frac{\partial^2 V}{\partial x_p \partial t_q} - \frac{\partial^2 V}{\partial x_q \partial t_p}. \tag{4}$$

It is easy to verify that when V is defined by an equation of type (3) the three partial differential equations of type

$$H_x^{p,q} = \pm iE_x^{p,q} \tag{5}$$

are satisfied, the upper or lower sign being taken according as the upper or lower sign is taken in (2).

A multiple wave function V which satisfies the three partial differential equations of type (5) will be called *right-handed* or *left-handed with respect to p and q* according as the upper or lower sign is taken. If, however, both $H^{p,q}$ and $E^{p,q}$ are zero it will be called *neutral with respect to p and q*. When a multiple wave function is either right-handed or neutral with respect to each pair of numbers p, q it will be called a *right-handed function* and will be denoted by the symbol V_+. A left-handed function is defined in a similar way and will be denoted by the symbol V_-. The function given by (3) is either right-handed or left-handed according as the upper or

---

* There are of course exceptions to this rule as for instance when

$$V = [(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2 + (z_n - z_{n-1})^2 - (t_n - t_{n-1})^2]^{-1}.$$

† If we regard the t's as time variables the units must be chosen so that the velocity of propagation of the waves is represented by unity.

lower sign is taken in (2); it is neutral with respect to p and q when f satisfies the partial differential equation

$$\frac{\partial^2 f}{\partial \alpha_p \partial \beta_q} = \frac{\partial^2 f}{\partial \alpha_q \partial \beta_p}. \tag{6}$$

A multiple wave function may of course be neutral with respect to one pair of numbers p, q and either right-handed or left-handed with respect to another pair; it is only *completely neutral* when all the vectors $H^{p,q}$ and $E^{p,q}$ are null. The function V represented by (3) is thus *completely neutral* when all the partial differential equations of type (6) are satisfied. A completely neutral function may be denoted by the symbol V_0. In general, of course, a multiple wave function does not possess the properties of left-handedness, right-handedness and neutrality, because it is of the form $V = V_+ + V_-$. It may happen that V_- is neutral with respect to p and q, while V_+ is not; in this case the function V is right-handed with respect to p and q; moreover V_+ may be neutral with respect to r and s while V_- is not and then V is left-handed with respect to r and s. Thus a multiple wave function may be partially right-handed, partially left-handed, and partially neutral.

2. When the point (x_n, y_n, z_n, t_n) coincides with $(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1})$ we shall suppose that the function $V^{(n)}$ reduces to a function which we shall denote by the symbol $V^{(n-1)}$. We may thus form a series of multiple wave functions

$$V_1, V_2, \cdots, V_{n-1}, V_n, \cdots, \tag{7}$$

possessing the property that, when the n points (x_p, y_p, z_p, t_p) coincide in succession, V_n reduces to V_{n-1}, V_{n-1} to V_{n-2}, and so on, the last function V_1 being a simple wave function. Instead of considering the process of reduction it is more interesting to consider the process of the development of a multiple wave function V_n from a simple wave function V_1. There is perhaps a slight analogy between this and the process of development of an organism from a single cell by repeated division. This analogy at once suggests the interesting problem to find a function V_1 and a method of development such that a certain characteristic property is preserved in the transition from V_{n-1} to V_n. This problem will be put on one side for the present and we shall use our analogy simply to form a convenient nomenclature.

We shall regard V as the characteristic function of an 'organism' and the point x_p, y_p, z_p, t_p as associated with a 'cell' (p) belonging to the organism. We see from (4) that a vector field ($H^{p,q}$, $E^{p,q}$) is associated with each pair of cells of the organism, and it is easy to verify that Maxwell's equations

$$
(8)\qquad
\begin{aligned}
&\frac{\partial H_z^{p,q}}{\partial y_s} - \frac{\partial H_y^{p,q}}{\partial z_s} = \frac{\partial E_x^{p,q}}{\partial t_s}, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\frac{\partial E_x^{p,q}}{\partial x_s} + \frac{\partial E_y^{p,q}}{\partial y_s} + \frac{\partial E_z^{p,q}}{\partial z_s} = 0 \\
&\frac{\partial E_x^{p,q}}{\partial y_s} - \frac{\partial E_y^{p,q}}{\partial z_s} = -\frac{\partial H_x^{p,q}}{\partial t_s}, \\
&\frac{\partial H_x^{p,q}}{\partial x_s} + \frac{\partial H_y^{p,q}}{\partial y_s} + \frac{\partial H_z^{p,q}}{\partial z_s} = 0 \\
&\qquad\qquad\qquad\qquad\qquad (s = 1, 2, \cdots, n)
\end{aligned}
$$

are satisfied for each set of variables x_s, y_s, z_s, t_s provided V can be represented as the sum of two integrals of type (3), one of which is right-handed and the other left-handed.

If now we take the real parts of the vectors $H^{p,q}$, $E^{p,q}$ we see that an electromagnetic field can be associated with a pair of cells (p) (q) except when the characteristic function V is neutral with respect to these two cells.

3. Let us now write $\xi_p = ix_p - y_p$, $\eta_p = ix_p + y_p$, $\sigma_p = z_p - t_p$, $\tau_p = z_p + t_p$ and expand the integral (3) by Taylor's theorem and Fourier's theorem; we then obtain a formal expansion of type

$$
(9)\qquad
\begin{aligned}
V_+ = \Sigma\, &\frac{\xi_1^{\mu_1}\xi_2^{\mu_2}\cdots\xi_n^{\mu_n}\eta_1^{\nu_1}\eta_2^{\nu_2}\cdots\eta_n^{\nu_n}}{\mu_1!\,\mu_2!\cdots\mu_n!\,\nu_1!\,\nu_2!\cdots\nu_n!} \\
&\times \frac{\partial^{\mu_1+\mu_2+\cdots+\mu_n+\nu_1+\nu_2+\cdots+\nu_n}}{\partial\sigma_1^{\mu_1}\partial\sigma_2^{\mu_2}\cdots\partial\sigma_n^{\mu_n}\partial\tau_1^{\nu_1}\cdots\partial\tau_n^{\nu_n}} \\
F(&\sigma_1, \sigma_2, \cdots, \sigma_n; \tau_1, \tau_2, \cdots, \tau_n; \\
&\mu_1 + \mu_2 + \cdots + \mu_n - \nu_1 - \nu_2 - \cdots - \nu_n).
\end{aligned}
$$

The expansion for V_- is of a similar type except that the positions of the variables σ and τ are interchanged and the function F is generally different.

If we assume that a right-handed multiple wave function can be expanded by Taylor's theorem in a series of ascending powers of $\xi_1, \xi_2, \cdots, \xi_n; \eta_1, \eta_2, \cdots, \eta_n$, then when we substitute this series in the partial differential equations (1) and (5) and equate the coefficients of the different powers of the ξ's and η's to zero we find that the series must necessarily have the form (9). If we limit the function V_+ to be a polynomial in the ξ's and η's, so as to avoid questions of convergence, we see from the form of the series that it can be expressed in the form (3). Similarly it can be shown that a left-handed multiple wave function which is a polynomial in the ξ's and η's can be expressed in the form (3) provided we take the lower signs in (2).

It follows from a theorem given in a former paper* that if the quantities $u_1, u_2, \cdots, u_n; v_1, v_2, \cdots, v_n$ are defined by the equations

$$\sigma_p = u_p + \xi_p\theta(u_1, u_2, \cdots, u_n; v_1, v_2, \cdots, v_n),$$
$$(10) \qquad \tau_p = v_p + \frac{\eta_p}{\theta}, \qquad (p = 1, 2, \cdots, n)$$

then the function

$$(11) \qquad V = \frac{\partial(u_1, u_2, \cdots, u_n; v_1, v_2, \cdots, v_n)}{\partial(\sigma_1, \sigma_2, \cdots, \sigma_n; \tau_1, \tau_2, \cdots, \tau_n)} f(u_1, u_2, \cdots, u_n; v_1, v_2, \cdots, v_n)$$

is a right-handed multiple wave function, θ and f being arbitrary functions of the $2n$ parameters $u_1, u_2, \cdots, u_n; v_1, v_2, \cdots, v_n$. If we expand this function in powers of the ξ's and η's using the generalized Darboux theorem† we obtain a series of type (9) in which

$$(12) \qquad F = f\theta^{\mu_1+\mu_2+\cdots\mu_n-\nu_1-\nu_2\cdots-\nu_n}.$$

To obtain the corresponding left-handed multiple wave function we must interchange the places of σ and τ.

4. Let us now consider the multiple wave functions which are homogeneous polynomials of degrees $m_1, m_2, \cdots, m_n$ with respect to the cells (1), (2), $\cdots$, (n) respectively. Since a

* Loc. cit.

† G. Darboux, *Comptes Rendus*, vol. 68, p. 324. Hermite, Cours d'Analyse. 4th edition, p. 182. See also T. J. Stieltjes, *Ann. d. l'École Normale* (3), vol. 2 (1885), p. 93. H. Poincaré, *Acta Mathematica*, vol. 9. K. de Fériet, Thèse, Paris, Gauthier-Villars (1915).

right-handed polynomial of this type can be expressed in the form (3) it follows that the function f must be of degrees m_1, $m_2, \cdots, m_n$ in the pairs of variables $\alpha_1, \beta_1 e^{i\omega}; \alpha_2, \beta_2 e^{i\omega}; \cdots; \alpha_n, \beta_n e^{i\omega}$ respectively and a polynomial of degree $m_1 + m_2 + \cdots + m_n$ in $e^{-i\omega}$. The number of arbitrary constants in the most general expression of this kind is

$$(13) \quad N_+ = (m_1 + 1)(m_2 + 1) \cdots (m_n + 1)(m_1 + m_2 + \cdots + m_n + 1),$$

hence we may conclude that there are N_+ linearly independent multiple wave functions which are right-handed homogeneous polynomials of degrees $m_1, m_2, \cdots, m_n$ with respect to the different cells. The number of linearly independent left-handed polynomials is represented by the same number.

To find the number of completely neutral polynomials of a given type we proceed as follows: Adopting a generalization of a method used by Cayley* we may derive one multiple wave function from another by operating on the latter any number of times with operators of type

$$(14) \qquad x_p \frac{\partial}{\partial x_q} + y_p \frac{\partial}{\partial y_q} + z_p \frac{\partial}{\partial z_q} + t_p \frac{\partial}{\partial t_q}.$$

This operator does not alter the character of the function relative to the cells p, q. An operator of the type

$$(15) \qquad y_p \frac{\partial}{\partial z_q} - z_p \frac{\partial}{\partial y_q} - i x_p \frac{\partial}{\partial t_q} + i t_p \frac{\partial}{\partial x_q}$$

gives a new right-handed multiple wave function when it operates on a multiple wave function which is either right-handed or neutral. So in this case the neutrality is lost.

We shall now show that the equation

$$(16) \qquad x_p \frac{\partial V}{\partial x_q} + y_p \frac{\partial V}{\partial y_q} + z_p \frac{\partial V}{\partial z_q} + t_p \frac{\partial V}{\partial t_q} = 0$$

is incompatible with the conditions of neutrality $E^{p,q} = 0$, $H^{p,q} = 0$, when V is a homogeneous polynomial of degrees $m_1, m_2, \cdots, m_n$ with respect to the different cells.

If we differentiate (16) with respect to x_p we find that

* *Liouville's Journal*, vol. 13 (1848), p. 275; Collected Papers, vol. 1, p. 397.

$$\frac{\partial V}{\partial x_q} + x_p \frac{\partial^2 V}{\partial x_p \partial x_q} + y_p \frac{\partial^2 V}{\partial x_p \partial y_q} + z_p \frac{\partial^2 V}{\partial x_p \partial z_q} + t_p \frac{\partial^2 V}{\partial x_p \partial t_q} = 0$$

or, since V is neutral,

$$\frac{\partial V}{\partial x_q} + x_p \frac{\partial^2 V}{\partial x_p \partial x_q} + y_p \frac{\partial^2 V}{\partial y_p \partial x_q} + z_p \frac{\partial^2 V}{\partial z_p \partial x_q} + t_p \frac{\partial^2 V}{\partial t_p \partial x_q} = 0.$$

Since V is homogeneous this equation reduces to

$$(1 + m_p) \frac{\partial V}{\partial x_q} = 0$$

and so (16) would imply that all the derivatives of V with respect to the variables x_q, y_q, z_q, t_q are zero. If then we suppose that V is not independent of these variables we may conclude that equation (16) is impossible.

It is now clear that by means of successive operations of type (14) we may derive a simple wave function of degree $m_1 + m_2 + \cdots + m_n$ from each completely neutral multiple wave function of degrees $m_1, m_2, \cdots, m_n$, and that, conversely, we may derive a completely neutral multiple wave function of degrees $m_1, m_2, \cdots, m_n$ from each simple wave function of degree $m_1 + m_2 + \cdots + m_n$. Hence it follows that the number of linearly independent polynomials of each type is the same and in the case of the simple wave function this number is known to be*

$$(m_1 + m_2 + \cdots + m_n + 1)^2.$$

Denoting this number by N_0, we can say that the number of linearly independent neutral polynomials of degrees $m_1, m_2, \cdots, m_n$ respectively with regard to the different cells is N_0. Since the neutral polynomials are included among both the right-handed and left-handed polynomials, we can expect that the total number of linearly independent multiple wave functions which are homogeneous polynomials of degrees $m_1, m_2, \cdots, m_n$ will be represented by $2N_+ - N_0$.

JOHNS HOPKINS UNIVERSITY,
BALTIMORE, MD.,
December 28, 1915.

---

* This follows at once from (13). See also Heine, Handbuch der Kugelfunctionen (1878), p. 472.

CHANGING SURFACE TO VOLUME INTEGRALS.

BY PROFESSOR E. B. WILSON.

(Read before the American Mathematical Society, February 26, 1916.)

THE note of Dr. Poor on "Transformation theorems in the theory of the linear vector function" in this BULLETIN, January, 1916, page 174, raises the question: Why not make the work short by using other methods?

The equation* $\int d\boldsymbol{S}(\quad) = -\int d\tau\nabla(\quad)$ is an obvious identity because

$$\int\int \boldsymbol{i}dydz(\quad) = -\int\int\int \boldsymbol{i}dydzdx\frac{\partial}{\partial x}(\quad)$$

is merely a partial integration.

If Φ be a linear vector function,

$$\nabla(\Phi\cdot\boldsymbol{u}) = \nabla_\Phi(\Phi\cdot\boldsymbol{u}) + \nabla_{\mathbf{u}}(\Phi\cdot\boldsymbol{u}) = -\nabla_M(\Phi\cdot\boldsymbol{u}) + \nabla\boldsymbol{u}\cdot\Phi_C,$$

where the subscripts Φ and $\boldsymbol{u}$ mean that the differentiation affects only the function indicated and the subscript M means that the differentiation is with respect to the point M of which $\boldsymbol{u}$ is independent (other differentiations are with respect to P). Hence, integrating with no sign, with dot, and with cross,

$$\int d\boldsymbol{S}\Phi\cdot\boldsymbol{u} = \int d\tau\nabla_M(\Phi\cdot\boldsymbol{u}) - \int d\tau\nabla\boldsymbol{u}\cdot\Phi_C, \qquad \text{Theorem 3,}$$

$$\int d\boldsymbol{S}\cdot\Phi\cdot\boldsymbol{u} = \int d\tau\nabla_M\cdot(\Phi\cdot\boldsymbol{u}) - \int d\tau\nabla\boldsymbol{u} : \Phi, \qquad \text{Theorem 2,}$$

$$\int d\boldsymbol{S}\times\Phi\cdot\boldsymbol{u} = \int d\tau\nabla_M\times(\Phi\cdot\boldsymbol{u}) - \int d\tau(\nabla u\cdot\Phi_C)_\times, \qquad \text{Theorem 1.}$$

Next if $\Phi\cdot d\Psi = d\Psi\cdot\Phi$, then $d(\Phi\cdot\Psi) = d\Phi\cdot\Psi + d\Psi\cdot\Phi$ and $\nabla(\Phi\cdot\Psi) = \nabla\Phi\cdot\Psi + \nabla\Psi\cdot\Phi$. Hence on integrating, we have

$$\int d\boldsymbol{S}\Phi\cdot\Psi = -\int d\tau\nabla\Phi\cdot\Psi - \int d\tau\nabla\Psi\cdot\Phi, \qquad \text{not given,}$$

* Reference may be made to my review, "The unification of vectorial notations," this BULLETIN, vol. 16, May, 1910, p. 428, where I use $d\mathrm{S}$ as an exterior normal instead of an interior normal as here.

that a variable always increasing and always less than some fixed number will have a limit (page 29). The author explains his view very clearly on page 33, but we think that this subject can be made quite simple, so much of it as is necessary for the proofs of a few fundamental theorems, and that it is well worth the time that might have to be taken from more advanced technical study. Even students who do not go on with mathematics will know better the mathematics that they do get, and will be better able to use it, if they understand better the foundations on which it rests.

Extreme care is shown in the accuracy of the proofs, yet there are certain forms of expression which might lead the immature student into loose ideas as to the necessity of always having a proof. The words "obvious" and "obviously" can usually be cut out without any loss whatever, and when they occur they are apt to leave the impression that some things are to be taken in mathematics as obvious. We find that "the theorem mentioned above is self-evident and requires no proof," that the reader "knows" some things (page 7), and that the discussion given of the law of the mean (page 143), although not an analytic proof, will make the reader "feel in the most convincing manner" that the law is true. Sometimes also a theorem is true because "the figure shows it" (see, for example, pages 58 and 139). The student should clearly understand that such considerations can form no part of a logical system.

These remarks do not apply to the work as a whole, but only to a few sections. In general, the importance of rigorous proofs and accurate details is emphasized.

The author has an informal way of discussing many things that makes them very interesting. Instead of a collection of dry facts and formulas we see the subject as it develops. See, for example, his remarks on imaginary numbers (page 10), the treatment of the ϵ-notation (pages 32–34 and 132), differentiation and integration at the beginning of Chapter VI, the significance of Cauchy's integral theorems (page 215), Taylor's theorem (page 223), the σ-function (page 359), and the ϑ-functions with zero arguments (page 431).

The book begins with a historical sketch of two pages. We should like to have seen a fuller account of the early history of this subject. A few historical remarks scattered through the book give a hint as to how interesting such an

account would be (see pages 23, 91, 289, 300, 395, 402, 410, 412, 418, 423, 425–426, 454–455). The names of Abel, Cauchy, Gauss, Jacobi, Legendre, and Weierstrass occur frequently, and several others are mentioned, but these names would mean much more if the student knew something of the men. In only a single case (pages 395 and 401) is there a reference to any of their writings, and in the entire book there are only seven specific references to other books or publications.

A much fuller index and a table of formulas would have been very useful, and the number of cross references could have been increased to great advantage.

The first chapter gives the usual account of the representation of complex numbers by points in a plane. There is no mention in the book of the linear function except once on page 88, nor of Riemann surfaces. We are accustomed to the early introduction of these topics, but the author has been able to do without them, and regards other subjects as more important for students who do not intend to specialize in mathematics.

Two chapters deal respectively with real and complex series, but the first of these two chapters is concerned chiefly with questions of convergence, while Chapter III takes up the various properties of series, operations with series, power series, and double series. The author explains very carefully the notion of limits (page 33), and introduces the admirable notation so freely used in his Functions of a Real Variable,

$$\text{“}\epsilon > 0,\ m,\ | c - c_n | < \epsilon,\ n > m.\text{”}$$

We should like to call attention also to his excellent treatment of the "associative and commutative" properties of series (pages 64–71), and of "row and column series" (pages 80–84). One theorem on the removal of parentheses (the second case at the bottom of page 67) might be made a little more general by assuming, not that A is a positive term series, but that the terms in the parentheses are all positive while the parentheses may have either sign. This, indeed, is the theorem required on page 70.

These two chapters are followed by a chapter on the functions employed in elementary mathematics. A third of this chapter (about 13 pages) comprises all that we have on algebraic functions, the book being devoted almost entirely

to one-valued or uniform functions. The rest of the chapter is a study of the exponential, circular, and logarithmic functions, starting from their series developments.

Chapter V is on real variables, being a résumé of some parts of the calculus and an account of curvilinear and surface integrals. It is chiefly in this chapter that the author appeals to intuition, or bases his theorems on geometrical considerations that do not constitute proofs. As we have already explained we think that a more rigorous treatment would not have been so very much more difficult. There are some interesting physical applications at the end of the chapter, to work, potential, electric current, and Stokes's theorem.*

Chapter V prepares the way for the study of complex differentiation and integration in Chapter VI. The Cauchy-Riemann equations are given, the theorem on conformal representation by means of a function having a derivative, and the properties of integrals of such functions. Steady (that is, uniform) convergence is also taken up, and the integration and differentiation of series. The treatment of all these subjects is very clear and simple.

Then we have a chapter on analytic functions, with Cauchy's theorems, Taylor's theorem, Laurent's theorem, and Fourier's development; a chapter on infinite products leading to Weierstrass's theorem (Mittag-Leffler's theorem not being mentioned); and a chapter applying these theorems to the study of the Beta and Gamma functions, and giving a somewhat difficult account of asymptotic expansions. The subject of analytic continuation as presented in the first of these three chapters is interesting. This term is used here in a slightly broader sense than usual; namely, to denote the process of finding the value of an analytic function at a point z when its values are known along some piece of a curve or line (page 225). In the chapter on infinite products† the sine

---

* On page 156 the set of equations immediately following (5) is parenthetical, put in for the purpose of deriving formulas to be applied to (5). It would have been clearer if these equations had been put in as a footnote, or printed in small type.

† On page 267 an infinite product is said to be convergent when the limit of the product of the first n factors is finite and not zero, or "when one of the factors is zero." This is not the usual definition and with this definition the theorems which follow are not all true.

On page 280, in considering the associative character of infinite products the author omits entirely the question of removing parentheses, which he treats very carefully in the case of infinite series (page 67).

and cosine products and associated series are given. It is always a delight to the student to find so many new and wonderful properties of the long familiar functions of trigonometry.

The relation of the modern function theory to the study of elliptic functions is about the same as the relation of analytic geometry to the study of conic sections. There are three chapters (X–XII) on the elliptic functions, the first on the functions of Weierstrass, the second on the functions of Legendre and Jacobi, and the third on the ϑ-functions and the relations of the two systems of the preceding chapters. As usual the functions of Weierstrass are developed from the point of view of their periodicity and the functions of Legendre and Jacobi from the elliptic integrals. There is a certain amount of duplication in the two systems, but both are needed, one being better adapted to some applications and the other to other applications. The ϑ-functions are the simplest functions in the older system and it is often convenient to express the other functions in terms of them. They correspond to the σ-functions of Weierstrass, and the exact relation between the two systems is obtained from the study of these two simplest types. We have in these three chapters a very good introduction to the elliptic functions with plenty of detail to work out and one or two applications. More applications would have been welcome.

Finally, three chapters (XIII–XV) on linear differential equations introduce us to some of the functions defined by equations of the second order. This subject is not usually considered in the text-books on the theory of functions, but these chapters put into our possession some of the functions which are most important in mathematical physics, and the methods used are only those developed in the preceding pages. The author's course in writing these chapters is in line with the practice of some writers on the calculus to add a chapter on the solution of differential equations as furnishing many valuable results from the methods previously studied, as well as the best possible kind of practice in these methods.

One or two details may be noticed. It would probably have conduced to clearness if the author on pages 41 and 284 had referred to the extended law of the mean or Taylor's theorem in finite form, instead of simply "the law of the mean" as he does. The proof at the bottom of page 245

requires that $c \neq 0$. For $c = 0$ the treatment would be somewhat different. On page 344, an elliptic function that has no pole in a parallelogram of periods is a constant, not because it "has no singular point anywhere in the infinite plane," but because it is "less than some G" in the parallelogram, and then less than this G everywhere in the plane. On page 364 (9), the coefficient of z^9 should be $- s_6/8 + s_2^2/32$. This shows that the law of the series is not as simple as the second and third terms would indicate. On page 429, line 1, we cannot get a_0 by putting $q = 0$ until we have shown that a_0 does not depend on q. The terms of the sine-series for $\vartheta_1(v)$ do not vanish when $v = m + n\omega$, as stated at the top of page 430, for $n \neq 0$. On page 473, if $m = l + \frac{1}{2}$ and $r = - m$, c_{2l+1} will not vanish but will be undetermined, and all the c's of odd index after this c will contain it as a factor. We can, however, arbitrarily give to this c the value zero and so get an integral independent of y_1 and without logarithms as desired. On pages 481–483 the formulas are not quite consistent as to the sign-factor $(- 1)^m$. The matter may be straightened out by putting this factor into equation (2) and taking it out of equation (3) and out of the expression for $\partial U_1/\partial s$. In equation (6) m should be used instead of n and $2^m\Pi(m)$ should be in the numerator and not in the denominator. In the expression for $\partial V_1/\partial s$ the first term should contain the factor 2^m, and the second term should be $C\, 2^{m-1}\Pi(m)\omega(m)J_m(x)$ instead of "$1/2\ C\ J_m(x)$." Also $\omega(0)$ should be put equal to 0 and not equal to 1. In (7) the logarithmic term should have the factor 2, "$\Pi(m)$" should be $\Pi(k)$, and at the end there should be the factor $(x/2)^{2k}$. On page 551, eight lines from the bottom the factor $|\gamma/a|^{s+1}$ should be $|t/a|^{s+1}$, and H/x^{s+1} should be Ht^{s+1}/x^{s+1}; in the integral in (10) $t^{\lambda-1}$ should be $t^{\lambda-s}$, and the limit of the integral will be $\Gamma(\lambda + s - 1)$.

The criticisms which we have offered do not reflect on the validity of the author's proofs. Many text-books present a view of mathematical reasoning that is entirely erroneous. Here we are taught the true spirit of modern rigor, and the student who studies this book properly should know what true mathematics is. This is far more important than mere verbal accuracy of detail.

These pages are right from the lecture-room; not always in smooth clear polished style, but full of life and an enthusiasm that carries us along as we read them. They are well adapted

to the classes in our American colleges, and we hope that they will be extensively used.

HENRY P. MANNING.

BROWN UNIVERSITY.

SHORTER NOTICES.

Analytic Geometry of Space. By VIRGIL SNYDER AND C. H. SISAM. New York, Henry Holt and Company, 1914. xi+289 pp.

THIS is one of the series of mathematical texts prepared under the general editorship of E. J. Townsend. The announced plan, however, of selecting as joint authors a mathematician and an engineer or physicist has not been adhered to in this case. As would be expected accordingly, the book will make its first and strongest appeal to the student of pure mathematics. If there is a single " practical problem " in the entire volume the reviewer has failed to discover it.

The authors are well fitted for their task since each is a specialist in the geometry of space and both are teachers of wide experience. Moreover their book possesses remarkable homogeneity of style and spirit, due possibly to the fact that the junior author was a pupil of the senior. At any rate, if there was any sharp division of labor the internal evidence is difficult to detect.

The book in size is an unpretentious volume of some 250 pages exclusive of the last chapter, and the modest preface states that it is intended as an introductory course. But even a casual examination will disclose an astonishing number and variety of topics, while a detailed reading emphatically confirms the first impression. Thus besides the usual equations of lines and planes and the metric formulas for angles and distances are introduced polar, cylindrical, and spherical coordinates, linear systems of planes, the notion of duality, homogeneous coordinates, and the plane at infinity, all in the first 37 pages! Surely this is information and ideas in a form sufficiently concentrated to stagger the average American undergraduate. It is only the very large number of excellent exercises (about 150 in the two chapters) which saves him from the otherwise inevitable confusion.

Indeed in our opinion the chief fault of the work as a *text book* is an excessive diversity of topics, many of which are treated very sketchily. As an extreme instance, the Kummer surface is introduced and dismissed with a half page. Whereas the worst blemish as a *book* is the frequent monotony of the expository style. This reaches a climax in Chapter VI in which the short, choppy, declarative sentences grow exceedingly tedious, the more because of the "vain repetitions."

While it is true, as indicated in the preface, that a knowledge of the usual elementary courses is all that is logically presupposed, it must be admitted that the first approach to such subjects as homogeneous coordinates, duality, linear systems, and the absolute is much simpler in the plane. And a student would be far better prepared for the present course after a thoroughgoing course in advanced plane analytic geometry. This statement applies to the entire book, but nowhere have we found the slightest hint of the natural method of attacking most topics in space, viz., that of generalization from the plane.

The book is divided broadly into two parts of approximately 100 and 150 pages respectively. The treatment in the first part is chiefly metric and "can be regarded as a first course not demanding more than 30 or 40 lessons." The subject matter of the first two chapters has already been indicated. The exposition is for the most part clear and concise but there are some obscurities of language. On page 1 it is stated that any point in space has three real coordinates. If one is to speak of real numbers, why not real points? "Orthogonal projection" is defined on page 3, while the theorems of page 4 use "projection" as the equivalent. The proof, page 12, that the three points are not collinear might have been made a little more explicit for the beginner who has not yet regarded the formulas of § 6 as parametric equations of the line. On page 21 "real value" is again associated with "point" without qualification. Article 23 is the first echo of the classical C. Smith, which has given analytic geometry of space such an awesome reputation in our colleges. Such puzzles are doubtless stimulating to the occasional American student, but the statement should be free from ambiguity. The wording of the paragraph is awkward and the figure is inappropriately lettered. PP' ought rather to be P_1P_2 and should be defined. In any case d refers to the other common secant.

The introduction of plane coordinates and elements at infinity is admirable, being simple, direct, and lucid. Moreover we commend the early discussion of these subjects (pages 31–35), since the notions are not intrinsically difficult and opportunity is afforded for the needed practice in their use. Not only is euclidean geometry greatly enriched, but the foundations are laid for the easy and natural transition to projective geometry.

Chapter IV is prefaced by a satisfactory treatment of geometric imaginaries. It is a matter of regret that imaginary elements are not recognized in our current texts on an equal footing with real,—that, e. g., such expressions as "point sphere" (§ 48), "the ellipse reduces to a point" (§ 56) should survive in the present text.

Chapter V is devoted to the sphere. We are glad to find a discussion of the absolute in this connection, for the subject belongs properly to metric geometry, though curiously enough it is usually included with projective.

In Chapter VI forms of quadric surfaces are studied from the standard equation by means of plane sections. This is a useful chapter, serving as an introduction to the following. Chapter VII contains an excellent treatment of the general equation of the second degree following the usual lines. Some of the important results are employed in § 75 to formulate a method of attack in discussing any particular quadric. This article would be improved pedagogically if it were amplified into a summary by including the criteria of §§ 66 and 73. The reduction of the general equation to the canonical forms and hence the complete classification of quadrics is effected with small labor and without actually applying the formulas for the transformation of coordinates.

The authors mention the definitions of page 76 among the features of Part I. In our opinion they are a bad feature unless the associated type of surface is indicated. What student would ever guess from the definition that "line of vertices" is the common line of two intersecting planes, or even that the quadric degenerates? Worse still is "plane of centers" and "plane of vertices" with the even greater specialization involved.

The second part of the book (Chapters IX–XIII) is devoted almost exclusively to the projective geometry of space. Many advanced students will find this part an attractive introduction

to the modern methods in the subject, while the variety of topics and the completeness of some of the results make it a valuable reference book as well.

Chapter IX is introductory and has to do with the tetrahedral coordinate system, duality, and the transformation of coordinates. The projective properties of quadric surfaces are developed in Chapter X and simplified forms of the general equation obtained for various reference systems.

A special feature of the entire book is the extensive space given to linear systems. Systems of planes and spheres are considered in Part I. The whole of Chapter XI, the longest in the book (40 pages), is occupied with linear systems of quadrics. An exhaustive classification of pencils of quadrics by the theory of elementary divisors is obtained. And this study is paralleled in Chapter XII by a complete classification of the types of projective transformations of space. Bundles (systems of three) are also examined at some length. Webs (systems of four) are considered briefly and the Weddle and Kummer surfaces defined by means of them.

The polar theory of surfaces is outlined in Chapter XIII. Space curves are introduced as intersections of surfaces and their properties deduced largely from this point of view. Fifteen pages are given to cubic and quartic curves, including a classification, metric in the case of the cubic since all space cubics are rational. The characteristic symbols for curves are likely to be confused (in § 186, e. g.) with those for pencils of quadrics, since they differ in some cases only by a comma. Parentheses might be used in one case instead of brackets.

The book closes with a chapter (23 pages) on metric differential geometry, after the modern fashion of appending a few pages on a subject itself requiring a treatise. Such supplements are necessarily fragmentary but it is surprising to see how much can be condensed into these introductory sketches.

A review would be incomplete without a statement regarding the exercises. It has been our fortune to see few books with such a wealth of good exercises. There is hardly a topic in all the incredible variety which is not amply illustrated, and they are suited to all classes and abilities of students. It is safe to say that any one who solves any considerable proportion of them must possess a pretty comprehensive grasp of the methods of analytic geometry as well as a very respectable store of its subject matter.

The following typographical errors have been noted:

P. 8, l. 4	for NP	read NP_2
P. 14, l. 23	" ON	" OM
P. 21, first eq. (18)	" b_2	" b_1
P. 28, l. 5	" D^2	" D_2
P. 63, last line	" $a^2\left(1-\frac{2}{c^2}\right)$	" $a^2\left(1-\frac{k^2}{c^2}\right)$
P. 64, eq. at bottom	" y	" z
P. 71, l. 4	" section	" sections
P. 72, § 64, eq. 2	" y^2/a^2	" y^2/b^2
P. 77, eq. (9)	interchange f and g	
P. 84, l. 17, second term	for κ	read κ^2
P. 85, last eq.	" f, g, h	" h, f, g
P. 92, l. 4	" (5)	" (6)
P. 97, l. 17	" y^3	" y^2
P. 101, mid. page	" $a\sqrt{}$	" $\pm a\sqrt{}$
P. 102, l. 5	" $c\sqrt{}$	" $\pm c\sqrt{}$
P. 103, l. 9	" hyperbolic	" parabolic
P. 115, l. 15	" $x)$	" (x)
P. 132, l. 2	" $A(xz)$	" $A(xy)$
P. 169, l. 2 (bottom)	" substantiated	" substituted
P. 177, l. 8	" poin	" point
P. 200, l. 17	" conic	" conics
P. 243, l. 9	" cubic	" quartic

R. M. WINGER.

Elementary Mathematical Analysis, a text-book for first year college students. By CHARLES S. SLICHTER, Professor of Applied Mathematics, University of Wisconsin. New York, McGraw-Hill Book Company, 1914. Price $2.50. xiv + 490 pp.

IN the older texts on pure mathematics the intellectual interest of the student in the subject was assumed, and the practical interest in applications was not given recognition. In many modern discussions of the place of mathematics in instruction the possibility of an intellectual interest in the subject, the possibility that a real need of reasoning, intelligent beings is satisfied by pure mathematics is denied and only that which ministers quite directly to the physical being is recognized. The present text, while it gives some attention to the intellectual side, places the real stress upon the applications to practical affairs, apparently justifying the study of mathematics because of its service to other sciences and to business.

Trigonometry, analytical geometry, and calculus have undoubtedly been made the fundamental mathematical studies

in college curricula because of the usefulness of these subjects in the sciences as well as because they are indispensable for further study in mathematics. However, like geometry and algebra in the high-school courses, these subjects have also been taught because they lend themselves to systematic and logical treatment. In analytical geometry, for example, we expect to find the fairly complete discussion of the general equation of the first degree and the similar discussion of the general equation of the second degree. The circle has commonly received analytical treatment because this procedure throws light upon the similar treatment of the other conic sections, enabling the student to comprehend easily the geometrical properties of the other conics as obtained by analytical methods. Entirely aside from the possibility of application to practical affairs the feeling has been that the student obtains, by these methods, some real appreciation of mathematics, of number, and of form. Further than this, even of those who have desired the study of trigonometry, analytical geometry, and calculus because of their applicability to science, many have felt that by the study of the elements of these subjects, without the complications introduced by physical data, the student would be able the better to handle these tools when confronted with a real problem.

Historically the study of the properties of the conic sections by the Greeks was entirely independent of mundane purposes. Yet this study did make possible the achievements of Kepler and Newton; these Greek studies prepared the way for the development of modern mathematics.

In this text the reader will look in vain for any systematic discussion of the straight line and the general equation of the first degree. The formulas for the distance between two points, for the area of a triangle formed by three points, and for the point of division of the line joining two points do not appear at all. The "point-slope" and the "two-point" formulas for the straight line appear towards the end of the work (pages 431–432). The circle and the conic sections are treated after curves of the form $y = ax^n$. While the general equation of the second degree receives consideration, this seems to be, according to a footnote, in some measure as a concession to correspondence courses. Tangents and normals receive scanty treatment.

The table of contents shows an entirely different order of

procedure from that to which we have grown accustomed. The chapter headings are as follows: I. Variables and functions of variables. II. Rectangular coordinates and the power function. III. The circle and the circle functions. IV. The ellipse and hyperbola. V. Single and simultaneous equations. VI. Permutations, combinations, the binomial theorem. VII. Progressions. VIII. The logarithmic and exponential functions. IX. Trigonometric equations and the solution of triangles. X. Waves. XI. Complex numbers. XII. Loci. XIII. The conic sections. XIV. Appendix—a review of secondary school algebra.

An immense amount of material is included within the book; the treatment of this material is not characterized by simplicity. In both of these respects the work is in striking contrast with modern high-school texts on algebra and geometry. Here the tendency has been to simplify by exclusion and to adapt the material presented in every way to the pupil. Any attempt to treat the freshman in college as a superior order of being, as compared with the high-school student, would seem to be doomed to failure.

While the attempt to introduce into the freshman course in mathematics some practical applications of the mathematical material is highly to be commended, to make the entire course center on the discussion of the highly technical is little short of ridiculous. Take as illustration the chapter on "Waves," which is not preceded by any discussion of the parabola and by only the briefest treatment of the ellipse and hyperbola. In this chapter we have extensive and intensive, for the freshman, study of simple harmonic motion, besides stationary waves with a number of problems about "seiches," compound harmonic motion, harmonic analysis, sinusoidal function, and connecting rod motion. What wonderful freshmen! Nothing that has ever been given anywhere to freshmen students of mathematics could be more impractical for the freshman than this material. Similarly, too, topics like the discharge of water over trapezoidal weirs, the capacity of smooth concrete flumes, the flow of water in clean cast-iron pipes have no place in freshman mathematics; fortunate is the engineering school whose seniors are able to discuss intelligently such problems.

The teacher of elementary college mathematics will find some valuable suggestions such as the "Illustrations from Science"

(pages 65–71), the use of logarithmic paper, and the proper treatment of physical data.

Typographical errors are numerous. Among other errors "the trajectory of the projectile of a German army bullet" (page 396) is particularly offensive. The statement (page 214) that the principle of logarithms "had been quite overlooked by mathematicians for many generations" is not correct, for the principle was known even to Archimedes and appeared and was discussed in books of the sixteenth century. The development of negative, fractional, and irrational numbers (page 355) is the logical one, and not from "the history or algebra." In the treatment of trigonometry the constant use of all six trigonometric functions would seem to be open to criticism. There appears also repeated emphasis upon rather trivial schemes for memorizing formulas and even the signs of ordinate and abscissa (or of sin α, cos α, and tan α).

Doubtless in the customary instruction in freshman mathematics too little attention has been paid to the functions $y = ax^n$, $y = a \sin mx$, and $y = k \cdot a^x$, and to the elementary applications of these functions and of the conic sections. Possibly in the future some way will be found to include in the freshman course, while preserving a logical treatment of the mathematical material, some applications which will be practical from the standpoint of the freshman. The present text does not appear to be successful either in logical treatment or in the presentation of practical material adapted to first-year students.

LOUIS C. KARPINSKI.

An Introduction to the Theory of Automorphic Functions. By LESTER R. FORD, M.A. (Harv.) G. Bell and Sons, Limited, London, 1915. viii+96 pp. Price 3s. 6d.

THIS is No. 6 of the Edinburgh Mathematical Tracts and has its origin in a series of lectures on automorphic functions given by Mr. Ford to the Mathematical Research Class of the University of Edinburgh during the spring term of 1915.

Mr. Ford has endeavored to bring out "the concepts and theorems on which the theory is formed, and to describe in less detail certain of its important developments." The tract is therefore conceived in the nature of an orientation rather than that of a treatise, and contains six chapters: I, Linear transformations; II, Groups of linear transforma-

tions; III, Automorphic functions; IV, The Riemannian-Schwarz triangle function; V, Non-euclidean geometry; VI, Uniformization.

In the chapter on linear transformations the proof, on pages 5–6, that the linear transformation is the most general one yielding a one-to-one correspondence between a z- and a z'-plane is incomplete. The most general function of this kind may be written in the form

$$z' = f(z) = \frac{A_m}{(z-g)^m} + \frac{A_{m-1}}{(z-g)^{m-1}} + \cdots + \frac{A_1}{z-g} + \phi(z),$$

where g is the only pole which may occur and where $\phi(z)$ is holomorphic in the entire z-plane. For, if ϕ were not holomorphic throughout (including $z = \infty$), $f(z)$ would have more than one place where $z' = \infty$, which is against the supposition of a one-to-one correspondence. $\phi(z)$ must therefore be a constant. The rest of the proof then follows as shown on page 6.

In the discussion of the regular solids, Chapter IV, the labeling of the regions in the stereographic projection of the octahedral group by the corresponding substitutions 1, S, T, and their products would greatly benefit the student. It should also be shown, by one example at least, how polyhedral functions based upon these groups may be constructed.

The connection between non-euclidean geometry and groups of linear substitutions may be established more explicitly by Poincaré's own method. Another model of a very clear, brief, and effective demonstration of this proposition may be found in "Die Idee der Riemannschen Fläche," by H. Weyl, pages 148–159.

It seems to me that the exceedingly important subject of "uniformization," even in a mathematical tract, should have received a fuller treatment. Reference to the elementary example of the uniformization of curves of deficiency 1 by elliptic functions would have added interest to this chapter.

The bibliography of automorphic functions at the end is a most valuable feature of the little book. We miss a reference to Gauss. See remarks by Fricke in Gauss's Collected Works, volume 8, pages 102, 103, volume 3, page 477.

But I suppose that desiderata of all kinds regarding an introduction to the vast subject of automorphic functions, limited to 96 pages, vary from person to person.

It must be said, however, that within this space Mr. Ford has succeeded well in the task which he has set for himself.

ARNOLD EMCH.

A Course in Interpolation and Numerical Integration for the Mathematical Laboratory. By DAVID GIBB. (Edinburgh Mathematical Tracts, No. 2.) London, G. Bell and Sons, 1915. viii+90 pp.

A Course in Fourier's Analysis and Periodogram Analysis for the Mathematical Laboratory. By G. A. CARSE and G. SHEARER. (Edinburgh Mathematical Tracts, No. 4.) London, G. Bell and Sons, 1915. viii+66 pp.

THESE two little volumes of the series edited by Professor Whittaker treat some of the more essential parts of the subjects of interpolation and numerical approximation, the first being devoted chiefly to the non-periodic case of polynomial interpolation, the second mainly to trigonometric interpolation in the representation of periodic functions. In the first volume, after a very brief introductory chapter on finite differences, Chapter II is devoted to the various standard interpolatory formulas of Lagrange, Newton, Stirling, etc., and closes with a brief account of numerical differentiation. Chapter III, on the construction and use of mathematical tables, is in part devoted to explaining in detail the application of the foregoing principles to direct and inverse interpolation, and in part to special methods for computing tables of logarithms. Finally Chapter IV is concerned with numerical integration.

The second volume begins with a chapter which gives in barest outline and quite without proofs the main facts which the practical man must know about Fourier's series. This chapter closes with Bessel's elegant deduction of a finite trigonometric sum which gives the best approximate representation of a function in the sense of the method of least squares when the values of the function at equally spaced points are known. It is the evaluation of the coefficients of these finite sums (not of Fourier's series) which is considered in Chapter II by various methods, chief among which are the systematized methods of computation devised by Runge in the cases of 12 and 24 ordinates. Certain graphical methods are also explained, but the instruments which effect this interpolation mechanically are explicitly excluded. Chapter III, which is entitled Periodogram Analysis, is devoted to a discussion of

the following problem: Some natural phenomenon is represented graphically by an undulating (but not periodic) curve. It is required to represent this curve, if possible, either completely or with a small irregular residuum, by the superposition of a number of simple harmonic curves, whose periods, phases, and amplitudes are all to be determined. Two different methods of treating this problem are here given, one of which goes back, in part, to Lagrange. Chapter IV opens with a very brief description of spherical harmonics and the development of arbitrary functions on a spherical surface which proceed according to them. The rest of it is devoted to a discussion of F. Neumann's method for the practical calculation of the coefficients in such developments.

The exposition of the purely formal sides of the subjects treated is clearly and attractively done. The task of compressing the preliminary theoretical matter into very brief space, without making it wholly unintelligible, is such a difficult one that one is inclined not to criticize, especially as these are not the essential parts of the books. One wishes, however, that a little more stress might have been laid on some aspects of interpolation which the writers would perhaps class as theoretical. The degree of accuracy attained by the various approximative formulas is hardly touched upon, the very serviceable remainder in Simpson's rule, for instance, being not even mentioned. It should surely have been pointed out that Bessel's formulas for trigonometric interpolation given at the close of Chapter II of the second volume under review contain undetermined parameters when the number of ordinates is less than the number, $2r + 1$, of coefficients. This is the case which is used in Chapter III, where the number of ordinates is even. Moreover, the reader is left in doubt whether the 12 and 24 ordinate formulas of Runge give *exact* coincidence with the desired values at the points in question; or, perhaps, if the reader has not passed beyond the stage of counting constants, he will not even be in doubt. That we do have exact coincidence might easily have been demonstrated, and should at least have been explicitly stated.

Throughout both volumes great stress is laid on actual numerical computation, substantial numerical examples being almost everywhere worked out and others left for the reader to carry through. This is the strong side of the books and

one of their characteristic features. In other ways also much valuable material has here been brought together for which persons wishing to learn or teach the subject will feel grateful to the authors.

MAXIME BÔCHER.

Ueber die analytische Fortsetzung des Potentials ins Innere der anziehenden Massen. Preisschriften gekrönt und herausgegeben von der Fürstlich Jablonowskischen Gesellschaft zu Leipzig. XLIV. By GUSTAV HERGLOTZ. Leibzig, Teubner, 1914. 52 pp.

THE problem for which this memoir is a solution was stated as follows: A given ellipse is transformed by the method of reciprocal radii into a certain oval. Consider a plane surface of homogeneous material (flat plate) bounded by such an oval. In the *Abhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften*, 1909, C. Neumann proved on the basis of the theory of the logarithmic potential that, so far as its action on exterior points is concerned, the surface in question can be replaced by a material line bounded by two mass points. Since it might be of interest to extend this result to space of three dimensions, the Society proposed the following question: In the theory of the newtonian potential what is the analogue of Neumann's theorem? That is, what can be said about the newtonian potential of a homogeneous solid bounded by the ovaloid which is obtained from an ellipsoid by the method of reciprocal radii?

The point of view adopted by the author is exhibited first in a generalization of the result obtained by Neumann for the case of the logarithmic potential. If the potential of a body for outside points can be represented as the potential of a suitable mass distribution within the body, then certainly it can be continued analytically inside the body up to the mass distribution. Conversely, if the potential can be continued analytically to points inside the attracting body and if the singular points of the potential which are encountered be enclosed by any surface (curve in the plane problem) F, then the potential, because it is regular outside F, can be represented as the potential of a mass distribution on F. In general the mass consists of a single and a double distribution, the density of which depends on the surface chosen.

The exact meaning of the analytic continuation of the

potential is apparent only by following the transformation by which the potential is expressed as a line integral of a function of a complex variable. The logarithmic potential of a flat plate of unit density upon an exterior particle at (ξ, η) is defined by

$$V(\xi, \eta) = \int\int \log \frac{1}{R} dxdy, \quad \{R^2 = (x - \xi)^2 + (y - \eta)^2\}.$$

The components of attraction are

$$X = \frac{\partial V}{\partial \xi} = \int\int \frac{x - \xi}{R^2} dxdy,$$

$$Y = \frac{\partial V}{\partial \eta} = \int\int \frac{y - \eta}{R^2} dxdy.$$

If we set $\zeta = \xi + i\eta$, $u = x + iy$, $v = x - iy$, and define Ω by

$$\Omega(\zeta) = X - iY,$$

then

$$\Omega(\zeta) = \int\int \frac{dxdy}{u - \zeta},$$

and

$$V(\xi, \eta) = \Re \int \Omega(\zeta) d\zeta,$$

where $\Re$ denotes the real part of the integral. If

$$G(x, y; \zeta) = \frac{v}{u - \zeta},$$

then, by Green's theorem,

$$\int\int \left(\frac{\partial G}{\partial x} + i\frac{\partial G}{\partial y}\right) dxdy = \int_C G(dy - idx),$$

where the integral in the second member is taken around the curve which forms the boundary of the plate. Since the first member is equal to 2Ω, this equation becomes

$$\Omega(\zeta) = \frac{1}{2i}\int_C \frac{v}{u - \zeta} du.$$

The denominator is single-valued and does not vanish

throughout the integration. The singularities must enter through v, which is determined as a function of u from the equation of the curve C. If this curve is algebraic the possible singularities are branch-points and poles, which occur at the foci of the curve. As defined by Plücker the ordinary foci of an algebraic curve are the real points on the tangents to the curve which pass through the circular points at infinity. If the curve includes the circular points then the tangents at these points furnish extraordinary foci. The function v has a branch-point at an ordinary focus and a pole at an extraordinary focus. The analytic continuation of the potential means the process of allowing the curve C to shrink without passing over a singular point. If the singular points are joined by a curve L then L represents the limiting form which C may take.

The author shows that it is possible to determine (1) the density of a simple distribution of matter along L, (2) the moment of a double distribution along L, and (3) the masses of particles situated at the extraordinary foci so that the potential shall be the same as the potential of the plate on an exterior point.

This result is a generalization of Neumann's statement. Some particularly simple theorems are deduced for certain curves consisting of two ovals. If one oval is allowed to shrink to a point or the two ovals allowed to coalesce into a loop one obtains the general inversion curve of the ellipse or hyperbola and every theorem goes directly into those discovered by Neumann.

The general treatment of the newtonian potential for algebraic surfaces is promised by the author in a later dissertation. In the present paper the work is confined to solids bounded by surfaces of revolution of the form

$$(x^2 + y^2 + z^2)^2 + 4C(x^2 + y^2) - 4Bz^2 + 4D = 0,$$
$$(D - B^2)(D - C^2) > 0.$$

These surfaces fall into six different types, and, as limiting cases, they include: for $D = 0$ the surface obtained by inversion of a surface of the second order, which is the proposed prize problem; for $D = B^2$ the torus considered by Bruns.

For homogeneous solids bounded by these surfaces of revolution the following conclusion is given: the potential

can be represented as the potential due to a mass arranged as (1) a simple distribution along a segment of the Z-axis, (2) a simple distribution over a circular plate in the XY-plane ($x^2 + y^2 \leqq a^2$), and (3) one or more mass-points at the ends of certain segments or a radial double distribution on the circular plate.

The density of the mass distribution in the XY-plane and on the Z-axis is explicitly determined for the various feasible cases.

W. R. LONGLEY.

Das Schachspiel, und seine strategischen Prinzipien. VON M. LANGE. Zweite Auflage. No. 281, Aus Natur und Geisteswelt. Leipzig, Teubner, 1914. 108 pp. Mark 1.25. With portraits of E. Lasker and Paul Morphy.

THIS little volume, with the portrait of a mathematician as frontispiece, is included in the announcement of the series in which it appears among the mathematical works. While the strictly mathematical treatment is, of necessity, slight yet the attempt is seriously made to present an introduction to chess based upon somewhat fundamental, and partly mathematical, principles. The work marks a distinct advance, in a pedagogical way, in the literature of chess.

LOUIS C. KARPINSKI.

A Course in Descriptive Geometry and Photogrammetry for the Mathematical Laboratory. By E. LINDSAY INCE. Edinburgh Mathematical Tracts, No. 1. London, E. Bell and Sons, 1915. viii+79 pages, 42 figures.

THIS little book makes no claim of being a treatise, but endeavors to present the important features of descriptive geometry in such a manner that one may be instructed rapidly in the general processes employed. A short introduction sketches the whole problem as treated by the methods of orthogonal double projection, perspective and plane projection. Only about twenty pages are devoted to the treatment of lines and planes, yet in this short space many of the standard problems are well discussed. In the chapter on the applications to curves and surfaces no general statements are found, no attempts being made to have the processes apply to other surfaces than cones, cylinders, and spheres. The mathe-

matical terms employed are used correctly, and empirical processes are designated as such. The chapter on perspective begins with the definite problem of drawing the picture of a cube in given position. This is followed by a very brief outline of the general theory, each step being developed directly from the preceding illustration. The entire process is then compared with the treatment of the same problem by the method of double orthogonal projection. The last few pages of this chapter proceed along lines similar to those in the opening chapters of Cremona's Projective Geometry, but are much more condensed. The last chapter in the book contains a short introduction to photogrammetry. The use of the art in military operations is attested, suggesting that the author's notes had been very recently revised. The fundamental problem is explained in detail, and a few refinements mentioned. Perhaps this is sufficient to comply with the avowed purpose of the Edinburgh Tracts, but to the reviewer it seems much too brief to be of greatest service. The page is attractively made up, the type very clear and not offensively prominent, and the figures well drawn. The book certainly succeeds in teaching the essential features of descriptive geometry in a remarkably small compass.

VIRGIL SNYDER.

NOTES.

THE thirty-seventh regular meeting of the Chicago Section, being the sixth regular meeting of the Society at Chicago, will be held at the University of Chicago on Friday and Saturday, April 21–22. The twenty-eighth regular meeting of the San Francisco Section will be held at the University of California on Saturday, April 22. A regular meeting of the Society will be held at Columbia University on Saturday, April 29.

THE April number (volume 38, number 2) of the *American Journal of Mathematics* contains: "On the classification and invariantive characterization of nilpotent algebras," by OLIVE C. HAZLETT; "Determination of the order of the groups of isomorphisms of the groups of order p^4, where p is a prime," by R. W. MARRIOTT; "Correspondences determined by the bitangents of a quartic," by J. R. CONNER; "Infinite groups

generated by conformal transformations of period two (involutions and symmetries)," by EDWARD KASNER; "On the solutions of linear homogeneous difference equations," by R. D. CARMICHAEL.

AT the meeting of the London mathematical society held February 11 the following papers were read: By J. H. GRACE: "Theorems on straight lines intersecting at right angles," and "The classification of rational approximations"; by Mrs. G. C. YOUNG: "Infinite derivatives"; by E. H. NEVILLE: "The bilinear curvature and other functions of independent directions on a surface"; by D. BRODETSKY: "The attraction of equiangular spirals."

The following papers were read at the meeting of March 9: By P. A. MACMAHON: "Some applications of general theorems of combinatory analysis;" by H. F. BAKER: "Mr. Grace's theorem on six lines with a common transversal;" by H. E. J. CURZON: "The integrals of a certain Riccati equation connected with Halphen's transformation;" by HILDA P. HUDSON: "A certain plane sextic;" by W. P. MILNE: "The construction of co-apolar triads on a cubic curve;" by J. BONDMAN: "The dynamical equations of the tides."

AT the meeting of the Edinburgh mathematical society on February 11 the following papers were read: By E. L. RICE: "On the continued fractions associated with the hypergeometric equation"; by A. MILNE: "Note on the Peano-Baker method of solving linear differential equations"; by D. GIBB: "On integral relations connected with the confluent hypergeometric function"; by E. M. HORSBURGH: "A simple form of integrometer."

THE Paris academy of sciences has announced the following prize problems in mathematics for the year 1917: The Francoeur prize (1000 fr.) for the most meritorious memoir in pure or applied mathematics; the Bordin prize (3000 fr.) for an improvement in some important point of the arithmetic theory of non-quadratic forms; the Poncelet prize (2000 fr.) for the French or foreign author of the most meritorious book or memoir on applied mathematics during the last ten years; the Vaillant prize (4000 fr.) for the determination and description of all surfaces which can in two ways be formed by the

displacement of an invariable curve; the Saintour prize (3000 fr.) for general mathematics; the Petit d'Ormay prize (10000 fr.) for the best contribution to pure or applied mathematics.

THE department of roads and canals of the technical school at Delft announces the following prize problem for the present year:

"An investigation is desired such that inaccuracies that appear in the calculation of the distribution of forces in a statically undetermined system may be lessened by the choice of an appropriate system of variables. Errors in the drawings and those arising by discarding too many places of decimals in numerical approximations should both be considered. Finally, a measure of the degree of accuracy secured should be devised." As literature, see in particular the prize memoir of J. Pirlet "Fehleruntersuchungen bei der Berechnung mehrfach statisch-unbestimmter Systeme," Aachen 1909 and various articles by the same author in *Der Eisenbau*, 1910–1915.

Competing memoirs should be sent, under the usual conditions, to Professor J. Nelemans, Delft, Holland, not later than October 31, 1916. The prize, a gold medal, will be awarded January 8, 1917.

The following university courses in mathematics are announced:

COLUMBIA UNIVERSITY (summer session, July 10–August 18).—By Professor M. W. HASKELL: Differential equations, five hours; Modern analytic geometry, five hours.—By Professor JAMES MACLAY: Theory of geometric constructions, five hours.—By Professor EDWARD KASNER: Theory of functions of a real variable, five hours.—By Professor W. B. FITE: Higher algebra, five hours.

CORNELL UNIVERSITY (summer session, July 6–August 16).—By Professor VIRGIL SNYDER: Geometric constructions for high-school teachers, five hours; Seminar in algebraic geometry.—By Professor W. A. HURWITZ: Mathematical analysis, five hours; Supplementary problems in algebra for high-school teachers, five hours; Seminar in integral equations.—By Professor F. W. OWENS: Projective geometry, five hours; Seminar in foundations of geometry.

UNIVERSITY OF ILLINOIS (summer session).—By Professor E. J. TOWNSEND: Advanced calculus (functions of two real variables), five hours.—By Dr. A. R. CRATHORNE: Calculus of variations, five hours.

UNIVERSITY OF WISCONSIN. The following courses in mathematics are announced for the summer session, June 26 to August 4: By Professor E. B. SKINNER: Differential geometry. Linear substitutions.—By Professor H. W. MARCH: Theoretical mechanics. Infinite series and products.—By Professor H. C. WOLFF: Probabilities and statistics.—By Professor W. W. HART: The teaching of mathematics. Courses in analytic geometry and the calculus are also offered.

COLLÈGE DE FRANCE.—By Professor G. HUMBERT: Theory of quadratic forms in its relation to the theory of groups.—By Professor J. HADAMARD: Analytic theory of prime numbers. Professor C. J. DE LA VALLÉE POUSSIN has been invited to give a series of conferences at the Collège.

UNIVERSITY OF STRASSBURG (summer semester).—By Professor F. SCHUR: Differential and integral calculus, II, three hours; Theory of quadric surfaces, two hours; Seminar, two hours.—By Professor J. WELLSTEIN: Linear differential equations, three hours; Axonometry and perspective, two hours.—By Professor M. SIMON: Non-euclidean geometry.—By Professor P. EPSTEIN: Foundations of analysis, two hours.—By Dr. O. SPEISER: Mechanics, two hours.

THE Gamble prize of Girton College, Cambridge, has been awarded to Mrs. W. H. YOUNG for her contributions to mathematics.

AT Smith College, Professor HARRIET R. COBB has been promoted to a full professorship of mathematics and Miss PAULINE SPERRY has been appointed assistant professor of mathematics.

DR. OLIVE C. HAZLETT, of Radcliffe College, has been appointed associate in mathematics at Bryn Mawr College.

DR. E. J. MILES, of the Sheffield Scientific School of Yale University, has been promoted to an assistant professorship of mathematics.

Dr. R. W. Burgess, of Cornell University, has been appointed instructor in mathematics in Brown University.

Dr. G. A. Pfeiffer, of Harvard University, has been appointed instructor in mathematics in Princeton University.

Dr. Edward Kircher and Mr. W. L. Hart have been appointed to the Benjamin Peirce instructorships at Harvard University for the year 1916–1917.

Mr. H. B. Nixon, instructor in mathematics in Gettysburg College, died March 30.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Ahrens (W.). Mathematiker-Anekdoten. Leipzig, Teubner, 1916. 8vo. 4+56 pp. Karton. M. 0.80

Baier (W.). Zur Polartheorie des Flächenbündels 2. Ordnung mit besonderer Berücksichtigung des Flächenbündels der Raumkurve 3. Ordnung. Rostock, 1914. 8vo. 71 pp.

Beutner (W.). Ueber die primitiven Gruppen in sechs Veränderlichen. Giessen, 1914. 8vo. 30 pp. M. 0.60

Bieri (A.). Geometrische Darstellung der elliptischen Integrale 1. und 2. Art. Bern, 1914. 8vo. 73 pp.

Bohr (H.) og Mollerup (J.). Laerbog i matematisk Analyse. Forelöbig udarbejdelse til Brug ved Forelaesninger paa Polyteknisk Laeranstalt. Afsnit 1. Kjöbenhavn, 1915. 8vo. 498 pp. (Autographed.) M. 13.20

Brandt (H.). Zur Komposition der quaternären quadratischen Formen. Strassburg, 1913. 4to. 26 pp. M. 2.00

Bürger (W.). Die Rekonstruktion der Urform aus einer vorgeschriebenen Kovariante. Strassburg, 1913. 8vo. 69 pp. M. 2.50

Burkhardt (H.). See Encyklopädie.

Castelnuovo (G.). See Encyklopädie.

Crantz (P.). Analytische Geometrie der Ebene zum Selbstunterricht. Leipzig, Teubner, 1915. 8vo. 5+93 pp. M. 1.00

Ebner (P.). Ueber eine reziproke Zuordnung von Kurven im Raume. Würzburg, 1914. 8vo. 42 pp.

Eckert (H.). Ueber zwei den Eulerschen Funktionen $\Gamma(p)$ und $B(p, q)$ ähnliche Doppelintegrale. Leipzig, 1914. 8vo. 80 pp. M. 2.00

Encyklopädie der mathematischen Wissenschaften. Band II 1, Heft 7–8: H. Burkhardt, Trigonometrische Reihen und Integrale bis etwa 1850. Leipzig, Teubner, 1914–1915. Gr. 8vo. Pp. 819–1154+1155–1354. M. 10.50+6.20

——. Band III 2, Heft 8: G. Castelnuovo und F. Enriques, Grundeigenschaften der algebraischen Flächen vom Gesichtspunkte der birationalen Transformationen aus. Leipzig, Teubner, 1915. Gr. 8vo Pp. 635–768. M. 4.20

ENRIQUES (F.). See ENCYKLOPÄDIE.

FLORIN (E.). Untersuchung der Kurve, die den Polaren umhüllt wird, welche hinsichtlich einer Ellipse zu den Tangenten einer Parabel konjugiert sind. Münster, 1914. 8vo. 46 pp.

GRÄBNER (G.). Systeme von Geraden, welche bei der Fortbewegung des die Raumkurven begleitenden Dreikantes besondere Regelflächen erzeugen. Würzburg, 1913. 8vo. 40 pp.

HAHN (H.). Zur Theorie des Kegelschnittnetzes. Giessen, 1913. 8vo. 23 pp.

HILL (G. F.). The development of Arabic numerals in Europe, exhibited in sixty-four tables. Oxford, Clarendon Press, 1915. 4to. 125 pp. 12s. 6d.

HOLZBERGER (H.). Ueber das Verhalten von Potenzreihen mit zwei und drei Veränderlichen an der Konvergenzreihe. München, 1914. 8vo. 90 pp.

JAKS (E.). Beiträge zur Theorie des Kugelkreises. Königsberg, 1914. 8vo. 98 pp. M. 2.50

KLEINSCHRODT (K.). Ueber Kegelschnitts- und Kreisgeometrie. Jena, 1913. 8vo. 87 pp. M. 2.00

LANDIS (E. H.). See RICHARDSON (R. P.).

LEIBNIZ (G. W.). 9 Briefe an F. A. Hackmann. Herausgegeben von P. Ritter. Berlin (*Sitzb. Akad.*), 1915. Gr. 8vo. 17 pp. M. 1.00

LUTZ (E.). Untersuchungen über das 10-fach Brianchonsche Sechseck und das Pascalsche Sechseck im 10-fach Brianchonschen Sechseck. München, 1913. 4to. 48 pp.

MATHEWS (G. B.). Algebraic equations. 2d edition. Cambridge, University Press, 1915. 64 pp. 2s. 6d.

MILLER (G. A.). Historical introduction to mathematical literature. New York, Macmillan, 1916. 8vo. 14+302 pp. Cloth. $1.60

MOLLERUP (J.). See BOHR (H.).

MOSCHKOWITSCH (S.). Ueber Raumkurven, bei denen eine mit dem begleitenden Dreikant fest verbundene Gerade eine abwickelbare Fläche erzeugt. Jena, 1913. 8vo. 71 pp.

NEISS (F.). Rationale Dreiecke, Vierecke und Tetraeder. Leipzig, 1914. 8vo. 38 pp.

PETRI (M.). Systeme von Flächen zweiten Grades, die zu zwei gegebenen Flächen zweiten Grades in einer besonderen Beziehung stehen. Leipzig, 1914. 8vo. 55 pp. M. 1.20

RICHARDSON (R. P.) and LANDIS (E. H.). Fundamental conceptions of modern mathematics. Variables and quantities with a discussion of the general conception of functional relation. Chicago, Open Court, 1916. $1.25

RICHETTI (M.). Ueber diskontinuirliche und orthogonale Funktionensysteme. Zürich, 1914. 8vo. 33 pp.

RITTER (P.). See LEIBNIZ (G. W.).

S. J. Drei Gleichungen als Grundlage für einen Beweis des sogen. grossen Satzes von Fermat (Wolfskehlsche Preisaufgabe), allgemeinverständlich vorgeführt. Darmstadt, 1915. Gr. 8vo. 16 pp. M. 1.20

SCHOLLMEYER (G.). Ueber unendlich kleine Transformationen der Kurven konstanter Torsion und der Flächen konstanter Krümmung. Giessen, 1914. 82 pp. M. 2.00

STEIN (J.). Beiträge zur Matrizenrechnung mit Anwendungen auf die Relativitätstheorie. Tübingen, 1914. 8vo. 70 pp.

TÖPEL (R.). Zur Bestimmung der projektiven Transformationsgruppen des R_3, die keine ebene Mannigfaltigkeit invariant lassen. Tübingen, 1914. 8vo. 59 pp. M. 1.50

WILLIAMS (C. L.). The fourth-dimensional reaches of the Exposition. San Francisco, Paul Elder, 1915. 8vo. Paper.

II. ELEMENTARY MATHEMATICS.

CAJORI (F.). School arithmetic. Intermediate book. New York, Macmillan, 1915. 9+299 pp. 8vo. $0.40

CRATHORNE (A. R.). See RIETZ (H. L.).

DAVISON (C.). A first course of geometry. Cambridge, University Press, 1915. 89 pp. 1s. 6d.

LIETZMANN (W.). Geometrische Aufgabensammlung. Ausgabe B: für Realanstalten, Unterstufe. Leipzig, Teubner, 1916. Gr. 8vo. 8+239 pp. M. 2.80

MANCHESTER (R. E.). A brief course in algebra. Syracuse, C. W. Bardeen, 1915. 8vo. 198 pp.

MYERS (G. W.). See PALMER (C. I.).

PALMER (C. I.) and TAYLOR (D. P.). Plane geometry, edited by G. W. Myers. Chicago, Scott and Foresman, 1915. 8vo. 5+276 pp. $0.80

RIETZ (H. L.), CRATHORNE (A. R.) and TAYLOR (E. H.). School algebra. Second course. New York, Holt, 1915. 8vo. 10+235 pp. $0.75

TAYLOR (D. P.). See PALMER (C. I.).

TAYLOR (E. H.). See RIETZ (H. L.).

III. APPLIED MATHEMATICS.

ALBRECHT (B.). Ueber das Problem der Brachistochrone der Zentralbewegung für das Anziehungsgesetz $K = m \cdot r^n$. Halle, 1914. 8vo. 48 pp.+4 Tafeln.

ALEXANDER (T.) and THOMSON (A. W.). Elementary applied mechanics. 3d edition. London, Macmillan, 1916. 20+512 pp. 15s.

AUGHTIE (H.). Applied mechanics, first year. London, Routledge, 1916. 184 pp. 2s.

BERNEIS (B.). Bestimmung der spezifischen Wärme unvollkommener Gase nach der Durchströmungsmethode. Heidelberg, 1914. 8vo. 83 pp.

BIGELOW (F. H.). A meteorological treatise on the circulation and radiation in the atmospheres of the earth and of the sun. New York, Wiley, 1915. 8vo. 11+431 pp. $5.00

BRENTANO (J.). Ueber den Einfluss allseitigen hydrostatischen Druckes auf die elektrische Leitfähigkeit von Wismutdrähten ausserhalb und innerhalb des transversalen Magnetfeldes für Gleichstrom und Wechselstrom. München, 1914. 8vo. 77 pp.

BROWN (E. W.). See ENCYKLOPÄDIE.

BRUNN (A. v.). See ENCYKLOPÄDIE.

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome VI, volume 1, fascicule 1: N. Noirel, Triangulation géodésique; bases et nivellement; déviations de la verticale. D'après les articles allemands de P. Pizzetti. Paris, Gauthier-Villars et Leipzig, Teubner, 1915. Gr. 8vo. Pp. 1–224. M. 8.40

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band VI 2, Heft 5: E. W. Brown, Theorie des Erdmondes, übersetzt von A. v. Brunn; K. F. Sundman, Theorie der Planeten. Leipzig, Teubner, 1915. Gr. 8vo. Pp. 667–807. M. 4.40

FUSS (K.) und HENSOLD (G.). Lehrbuch der Physik. 14te vermehrte Auflage. Allgemeine Ausgabe. Freiburg, 1915. 8vo. M. 6.50

HALE (J. W. L.) Practical applied mathematics. New York, McGraw-Hill, 1915. 11+206 pp. $1.00

HELMHOLTZ (H.). Ueber die Erhaltung der Kraft (1847). Neue Ausgabe. 8tes Tausend. Leipzig, 1915. 8vo. 60 pp. M. 1.00

HENSOLD (G.). See FUSS (K.).

KLEIN (H. J.). Führer am Sternenhimmel für Freunde astronomischer Beobachtungen. 3te Auflage. Leipzig, 1915. 8vo. 480 pp. M. 10.00

KULLRICH (E.). Mathematisch-physikalische Tafeln. 2te Auflage. Leipzig, 1915. 8vo. 12 pp. M. 0.60

LANCHESTER (F. W.). Aircraft in warfare: the dawn of the fourth arm. London, Constable, 1916. 18+222 pp. 12s. 6d.

LEWIS (C. J.). Farm business arithmetic. Chicago, Heath, 1915. 8vo. 13+199 pp.

MANILIUS (M.). Astronomica. Edidit J. van Wageningen. Lipsius, 1915. 8vo. 26+196 pp. Leinenband. M. 4.50

NOIREL (N.). See ENCYCLOPÉDIE.

PIZZETTI (P.). See ENCYCLOPÉDIE.

STEVENS (J. S.). Theory of measurements. London, Constable, 1915. 6+81 pp. 6s.

SUNDMAN (K. F.). See ENCYKLOPÄDIE.

TAGGART (W. S.). Textile mechanics. London, Routledge, 1916. 7+117 pp. 2s.

THOMSON (A. W.). See ALEXANDER (T.).

WAGENINGEN (J. VAN). See MANILIUS (M.).

WESTERGAARD, (H.). Statistikens Theori i Grundrids. 2. omarbejdede Udgave. Kjöbenhavn, 1915. Gr. 8vo. 442 pp. M. 9.60

THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and eighty-second regular meeting of the Society was held in New York City on Saturday, February 26, 1916. The attendance at the morning and afternoon sessions included the following forty-three members:

Mr. J. W. Alexander, II, Mr. W. E. Anderson, Mr. D. R. Belcher, Dr. A. A. Bennett, Professor Joseph Bowden, Professor E. W. Brown, Professor J. G. Coffin, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor O. E. Glenn, Dr. T. H. Gronwall, Professor C. C. Grove, Professor H. E. Hawkes, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. Harry Langman, Mr. G. H. Light, Mr. P. H. Linehan, Professor E. J. Miles, Mr. G. W. Mullins, Mr. George Paaswell, Dr. P. R. Rider, Mr. J. F. Ritt, Dr. Caroline E. Seely, Dr. H. M. Sheffer, Professor L. P. Siceloff, Professor Mary E. Sinclair, Professor P. F. Smith, Mr. C. E. Van Orstrand, Professor Oswald Veblen, Mr. H. E. Webb, Dr. Mary E. Wells, Professor H. S. White, Mr. J. K. Whittemore, Miss E. C. Williams.

The President of the Society, Professor E. W. Brown, occupied the chair, being relieved by Professors Fine, Fiske, and White. The Council announced the election of the following persons to membership in the Society: Mr. L. E. Armstrong, Stevens Institute of Technology; Professor Grace M. Bareis, Ohio State University; Professor G. A. Chaney, Iowa State College; Mr. J. E. Davis, Pennsylvania State College; G. H. Hardy, M.A., Trinity College, Cambridge, England; Mr. Harry Langman, Metropolitan Life Insurance Company, New York City; Mr. E. D. Meacham, University of Oklahoma; Dr. A. L. Nelson, University of Michigan; Mr. Elmer Schuyler, Bay Ridge High School, Brooklyn, N. Y. Six applications for membership in the Society were received.

The Society has recently taken over the stock of the Chicago Papers and Boston Colloquium Lectures, heretofore in the hands of The Macmillan Company. All publications of the

Society, so far as in stock, are now obtainable directly from the main office. The New Haven Colloquium was published by the Yale University Press, and is sold by them.

The following papers were read at this meeting:

(1) Dr. T. H. Gronwall: "A functional equation in the kinetic theory of gases (second paper)."

(2) Dr. T. H. Gronwall: "On the zeros of the functions $P(z)$ and $Q(z)$ associated with the gamma function."

(3) Dr. T. H. Gronwall: "On the distortion in conformal representation."

(4) Dr. C. A. Fischer: "Equations involving the derivatives of a function of a surface."

(5) Professor E. W. Brown: "Note on the problem of three bodies."

(6) Dr. H. Bateman: "A certain system of linear partial differential equations."

(7) Dr. H. Bateman: "On multiple electromagnetic fields."

(8) Mr. A. R. Schweitzer: "On a new representation of a finite group."

(9) Mr. A. R. Schweitzer: "Definition of new categories of functional equations."

(10) Professor E. B. Wilson: "Critical speeds for flat disks in a normal wind: theory."

(11) Professor E. B. Wilson: "A mathematical table that contains chiefly zeros."

(12) Professor E. B. Wilson: "Changing surface to volume integrals."

(13) Dr. T. H. Gronwall: "Elastic stresses in an infinite solid with a spherical cavity."

(14) Dr. T. H. Gronwall: "On the influence of keyways on the stress distribution in cylindrical shafts."

(15) Professor O. E. Glenn: "The formal modular invariant theory of binary quantics."

(16) Professor O. E. Glenn: "The concomitant system of a conic and a bilinear connex."

(17) Dr. P. R. Rider: "Trigonometric functions for extremal triangles."

(18) Mr. H. S. Vandiver: "Symmetric functions of systems of elements in a finite algebra and their connection with Fermat's quotient and Bernoulli's numbers (second paper)."

(19) Mr. S. A. Joffe: "Calculation of eulerian numbers from central differences of zero."

The papers of Dr. Bateman, Mr. Schweitzer, Mr. Vandiver, and the first and third papers of Professor Wilson were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In a previous paper (*Annals of Mathematics*, September, 1915) Dr. Gronwall has shown that, $\varphi(\xi, \eta, \zeta)$ being the logarithm of the function defining the distribution of velocities, the functional equation for φ resulting from the Maxwell-Boltzmann fundamental theorem has as general solution the expression $a + b_1\xi + b_2\eta + b_3\zeta + c(\xi^2 + \eta^2 + \zeta^2)$, under the sole assumption of the continuity of φ for all finite values of the variables. In the present paper, it is shown that this condition may be replaced by the weaker one that φ shall be bounded in an arbitrarily small neighborhood of the origin.

2. Completing a result due to Bourguet, Haskins has recently shown (*Transactions*, 1915) that

$$P(z) = \sum_0^\infty \frac{(-1)^\nu}{\nu!} \frac{1}{z+\nu}$$

has one and only one zero in each of the intervals $-2m - \frac{3}{2} < z < -2m - 1$ and $-2m - 2 < z < -2m - \frac{3}{2}$ $(m = 2, 3, \cdots)$. The proof depends on the Budan-Fourier theorem. In the present paper, Dr. Gronwall shows, by a simpler method, that the real zeros of $P(z)$ lie one in each of the intervals $-2m - 1 - 6/(2m)! \leqq z \leqq -2m - 1 - 1/(2m)!$ and $-2m - 2 + 1/(2m+1)! \leqq z \leqq -2m - 2 + 6/(2m+1)!$, and furthermore that $P(z)$ has exactly four complex zeros, the real parts of which lie between $-\frac{17}{4}$ and $-\frac{3}{2}$, and the imaginary parts between $-\frac{11}{2}$ and $+\frac{11}{2}$. Finally it is shown, by an extremely simple application of Picard's theorem, that the entire function $Q(z) = \Gamma(z) - P(z)$ has an infinity of zeros.*

* It should be noted that Bourguet, *Comptes Rendus*, vol. 96 (1883), pp. 1487–1490, indicated approximations to the real zeros of $P(z)$, which are of essentially the same order as those of Dr. Gronwall. Bourguet also asserted that $P(z)$ can have at most four complex zeros. His proof, however, has been criticized by Nielsen, Handbuch der Theorie der Gammafunktion, p. 36, as inaccurate. It is worth noting, however, that by making use of the known analytic character of $P(z)$, and of the difference equation satisfied by it, Bourguet's proof can be made rigorous, and that then it gives, when taken in connection with Haskins's completion of Bourguet's theorem, the further fact that $P(z)$ has exactly four complex zeros. C. N. HASKINS.

3. In the present paper, Dr. Gronwall gives the exact determination of the bounds in Koebe's distortion theorem, which may now be stated in the following, and final, form:

"When the analytic function

$$w = z + a_2z^2 + \cdots + a_nz^n + \cdots$$

gives the conformal representation of the circle $|z| < 1$ on a simple region D in the w-plane, then we have for $|z| = r$ and $0 < r < 1$

$$\frac{1-r}{(1+r)^3} < \left|\frac{dw}{dz}\right| < \frac{1+r}{(1-r)^3},$$

$$\frac{r}{(1+r)^2} < |w| < \frac{r}{(1-r)^2},$$

except when $w = z/(1 - e^{\alpha i}z)^2$, in which case the upper and lower bounds are reached for $z = re^{-\alpha i}$ and $z = -re^{-\alpha i}$ respectively. When the region D is convex, we have

$$\frac{1}{(1+r)^2} < \left|\frac{dw}{dz}\right| < \frac{1}{(1-r)^2},$$

$$\frac{r}{1+r} < |w| < \frac{r}{1-r},$$

except for $w = z/(1 - e^{\alpha i}z)$, in which case the upper and lower bounds are reached as before."

It is also shown that the convexity bound, i. e. the upper bound of those values of r for which the map of $|z| < r$ is convex, exceeds $2 - \sqrt{3}$ except when $w = z/(1 - e^{\alpha i}z)^2$, where this value is reached. The above results are then extended in two directions, first by assuming the region D invariant for a rotation of $2\pi/n$ about the origin, and second, by assigning a priori the value of the second coefficient in w. A detailed abstract will appear in the *Comptes rendus*.

4. The functions considered in Dr. Fischer's paper depend on a surface and also on all of the values taken by a given function at points of the surface. Such a function has two partial fonctionnelle derivatives. The condition of integrability of an equation involving these two derivatives is found, and the characteristics are briefly discussed. Similar equa-

tions, involving functions of lines instead of surfaces, are discussed by Lévy in volume 37 of the *Rendiconti del Circolo Matematico di Palermo.*

5. Professor Brown's note gives a form for the equations of motion in the "restricted" case of the problem of three bodies when the first describes an elliptic orbit about the second, and the third is of zero mass. The curves of zero velocity are obtained and some other consequences deduced.

6. Dr. Bateman's first paper appeared in full in the April BULLETIN.

7. A vector field specified by two vectors H and E may be called a multiple electromagnetic field of rank n when the complex vector $M = H + iE$ satisfies the system of partial differential equations

$$\mathrm{rot}_p\, M = -\frac{i}{a}\frac{\partial M}{\partial t_p}, \quad \mathrm{div}_p\, M = 0 \quad (p = 1, 2, \cdots, n).$$

Dr. Bateman shows that each component of M is a right-handed multiple wave function of rank n and obtains various expressions for M by generalizing the solutions of Maxwell's equations which have been given by Hertz, Righi and others. A generalization is also obtained of a theorem due to Appell.

8. In previous communications* Mr. Schweitzer has indicated how any finite group can be represented formally by classes of functional equations derived from the generatrix: $f\{u_1, u_2, \cdots, u_{n+1}\} = \psi(x_1, x_2, \cdots, x_{n+1})$, where $u_i = f_i(t_1^{(i)}, t_2^{(i)}, \cdots, t_n^{(i)}, x_i)$ $(i = 1, 2, \cdots, n+1;\ n = 1, 2, 3, \cdots)$. Perhaps the simplest class of functional equations representing any finite group is the class† $\{E\left(\begin{smallmatrix}1, & 2, & \cdots, & [n+1]\\ i_1, & i_2, & \cdots, & i_{n+1}\end{smallmatrix}\right)\}$. The author obtains the general solution of this class of equations without referring to any particular substitution, and then proves the following theorem: Given any finite group G, there corresponds to this group a group of functional equations $E(G)$ such that (1) the groups G and $E(G)$ are simply isomorphic, (2) every equation of the group $E(G)$ possesses a solution.

* Cf. BULLETIN, vol. 22, pp. 4, 171.

† For the notation see BULLETIN, l. c., p. 4.

9. The categories of functional equations are defined by Mr. Schweitzer by means of the following generatrices:

$$f(u_1, u_2, \cdots, u_{n+1}) = \psi(x_1, x_2, \cdots, x_{n+1}) \\ + \phi(t_1^{(n+2)}, t_2^{(n+2)}, \cdots, t_n^{(n+2)}),$$

where $u_i = f_i(t_1^{(i)}, t_2^{(i)}, \cdots, t_n^{(i)}, x_i)$ $(i = 1, 2, \cdots, n+1;$ $n = 1, 2, \cdots)$ or analogous generatrices obtained by applying the following processes, either separately or in combination: (1) interchanging one of the u's with the symbol $\psi(x_1, x_2, \cdots, x_{n+1})$; (2) transposing homologously the x's on the left-hand side. From the preceding generatrices are derived functional equations by replacing some, none, or all of the t's with x's and assuming that the remaining t's (if any) with the same subscript are identical. When the function ϕ is identically zero, the functional equations thus defined include equations defined previously by the author. A simple example of one of the new types of functional equations is the relation

$$\lambda f\{f(t_1, t_2, \cdots, t_n, x_1), \cdots, f(t_1, t_2, \cdots, t_n, x_{n+1})\} \\ = \lambda f\{\alpha_1(x_{i_1}), \alpha_2(x_{i_2}), \cdots, \alpha_{n+1}(x_{i_{n+1}})\} + \phi(t_1, t_2, \cdots, t_n),$$

where the i_k range over $1, 2, \cdots, n+1$ and are distinct.

10. In calibrating the new wind tunnel at the Massachusetts Institute of Technology, Lieutenant J. C. Hunsaker, U. S. N., instructor in aeronautics, found a critical velocity for disks (2″ to 6″ in diameter) around 10 to 20 miles per hour. By considering the disks as ellipsoids of revolution and the air as a perfect liquid moving irrotationally, Professor Wilson shows that theoretically such a critical velocity, arising from incipient cavitation, should occur at velocities not much greater than those found experimentally. The sharp edges of the disks and the imperfections of the fluid and of its flow would probably lower the theoretical value found. The paper will appear in connection with Lieutenant Hunsaker's in *Smithsonian Miscellaneous Collections*, volume 62, number 4, article X.

11. Statistical results on the distribution of the different digits in the decimal development of an irrational number or in a tabulation of the values of a function appear not to be

numerous. Professor Wilson sets up an infinite series which defines a function that is perfectly healthy from the analytic viewpoint but has this peculiar statistical malady: The irrational value of the function for any terminating decimal value of the argument has in its infinite decimal development only an infinitesimal proportion of digits other than zero. This problem arose incidentally out of conversations with Professor Josiah Royce on probability and statistics.

12. Professor Wilson's note appeared in full in the April BULLETIN.

13. In the present paper, Dr. Gronwall determines the stress distribution in a solid with a spherical cavity, the dimensions of the solid being very large in comparison with the radius of the sphere, and the stress at a large distance from the cavity being assumed as pure tension or compression of constant magnitude T. It is shown that the maximum tensile stress at the surface of the cavity will be

$$\frac{39\lambda + 54\mu}{18\lambda + 28\mu} T,$$

λ and μ being the constants of elasticity. Thus the maximum stress is roughly twice as large as when there is no cavity, which corrects a statement by Larmor (*Philosophical Magazine*, 1892) that the influence of a cavity on the distribution of tensile stresses is negligible.

14. Dr. Gronwall considers a cylindrical shaft, into which is cut a keyway in the shape of a cylinder intersecting the former orthogonally, the shaft being subjected to torsional stress only. This shape of keyway is common enough in practice. The stress distribution is first determined by the methods of the theory of elasticity, and from this exact result an approximate formula is derived, adapted to practical use, and giving the ratio between the maximum stresses in the keyed and the full portions of the shaft as

$$\frac{2}{1 - 3b^2/a^2},$$

a and b being the radii of shaft and keyway respectively.

15. The first part of Professor Glenn's paper deals with methods of constructing formal invariants and covariants. A theorem is given on the determination of a covariant whose leading coefficient is any assigned seminvariant. Fundamental systems modulo 2 of first degree concomitants are derived for the quartic and the quintic.

In the second part it is shown that the higher forms are reducible in terms of their own first degree concomitants modulo 2. It is proved that if the systems of concomitants modulo 2 of the forms of orders 1, 2, 3 are all finite then the system of any form of order m is finite. It is then shown for the binary cubic modulo 2 that every covariant is quasi-reducible, on multiplication by a power of a first degree invariant, in terms of a set of fourteen concomitants.

16. By a ternary transvection process which Professor Glenn described in a paper communicated in February, 1915, he has derived the fundamental system of simultaneous concomitants of the set consisting of a singly quadric and a doubly linear form. The system furnished initially by the method contains above 100 forms. By various reduction processes he has reduced all but 67 forms. These are given as generalized transvectants and in terms of the symbols.

17. The triangles considered in Dr. Rider's paper are formed by extremal arcs. At one vertex the transversality condition of the calculus of variations is satisfied, thus making the triangles correspond to right-angled triangles in euclidean plane geometry. The trigonometric functions are defined as the limits of the ratios of the lengths of the sides as one side approaches zero, where by length is meant the value of a definite integral taken from one extremity of the arc to the other.

18. The present paper consists mainly of extensions along the lines indicated in Mr. Vandiver's first article under the same title (presented to the Society, April, 1913). A conjugate set in a finite algebra A is defined as a system of elements which are reproduced on multiplication of each element by some element of A. Symmetric functions formed by these sets are studied and for the case where A is represented by the incongruent residues modulo m, a rational integer, the

results obtained involve Fermat's quotient and Bernoulli's numbers.

19. In the *Quarterly Journal*, volume 45 (1914), pages 1–51, Professor J. W. L. Glaisher has calculated the first 27 eulerian numbers from certain recurring formulas and has shown that the method was especially advantageous when "curtate" formulas were employed. Mr. Joffe has verified Professor Glaisher's results and has extended the calculations to five more eulerian numbers by a different method based upon the formula

$$E_n = \sum_{m=0}^{n} (-1)^{m+n} e_{m,n},$$

where $e_{m,n}$ denotes $(1/2^m)\delta^{2m}0^{2n}$, and the quantities $\delta^{2m}0^{2n}$ are "central differences of zero." The successive terms $e_{m,n}$ are computed by a continuous process from the recurring formula

$$e_{m,n} = m[me_{m,n-1} + (m + \overline{m - 1})e_{m-1,n-1}],$$

and the final values of E_n are verified by the formula

$$E_n = \sum_{m=1}^{n-1} (-1)^{m+n+1}[(m+1)(m+2) - 1]e_{m,n-1}.$$

F. N. Cole,
Secretary.

ON A CONFIGURATION ON CERTAIN SURFACES.

BY PROFESSOR C. H. SISAM.

(Read before the American Mathematical Society, April 21, 1916.)

The surfaces here under consideration are rational and are generated by conics. They may be represented birationally on the plane in such a way that, to the plane or hyperplane sections of a given surface of the given type, correspond curves of order n having in common an $(n-2)$-fold point P_0 and Δ simple points $P_1, P_2, \cdots, P_\Delta$. We suppose further that $\Delta = 2k$, so that the surface is of even order, that $n > 3$ and that $k > 2$. For simplicity, we suppose that the fundamental points $P_0, P_1, P_2, \cdots, P_{2k}$ are in generic position.

The generating conics on the surface are determined by the

lines in the parametric plane through P_0. The $2k$ conics which correspond to the lines joining P_0 to $P_1, P_2, \cdots, P_{2k}$ are composite in such a way that the points of one component correspond to the points of the line P_0P_i, those of the other component to the directions through P_i.* We denote the former component right lines by the symbols $1, 2, \cdots, (2k)$, the latter by the symbols $1', 2', \cdots, (2k')$, respectively, so that the lines 1 and $1'$, 2 and $2'$, etc., constitute a composite generating conic.

The directrix curves of lowest order on the surface are 2^{2k-1} curves C^{n-2} of order $n-2$. These curves are determined as follows:

one, by the directions through P_0,
${}_{2k}C_2$ by right lines through two simple fundamental points,
${}_{2k}C_4$ by conics through P_0 and four simple fundamental points,
. . . .
one, by the curve of order k, which has a $(k-1)$-fold point at P_0 and passes through $P_1, P_2, \cdots, P_{2k}$,

wherein ${}_iC_j$ denotes the number of combinations of i things j at a time.

The C^{n-2} which is determined by directions through P_0 intersects the lines $1, 2, \cdots, (2k)$. We shall denote it by the composite symbol $1\ 2\ 3 \cdots (2k)$. If, in this symbol, we put, in all possible ways, primes on an even number of the component symbols we determine, for each of the remaining C^{n-2}, a symbol which indicates the lines on the surface that are intersected by it. Two such C^{n-2} which have 2σ component symbols unlike intersect in $\sigma - 1$ points as may be seen by counting the intersections, not at fundamental points, of the corresponding curves in the parametric plane. In particular, each C^{n-2} is intersected in $k-1$ points by a single C^{n-2} of the system.

We can choose sets of precisely $2k$ of the curves C^{n-2} in such a way that no two curves of the set intersect. In fact, if we birationally transform the parametric plane so that a given curve C_1^{n-2} corresponds to the directions at P_0, then the C^{n-2} which do not intersect C_1^{n-2} correspond to right lines through pairs of simple fundamental points. The C^{n-2} which correspond to the right lines joining a given simple fundamental point P_i to the remaining simple fundamental points form, with C_1^{n-2}, a set of the required type. Conversely,

* Clebsch, *Math. Ann.*, vol. 1, p. 266.

any given set can be represented in this way. Any $2k - 1$ of the C^{n-2} of such a set determine a unique right line on the surface which intersects them all. Of the $2k$ lines determined in this way by a set, an odd number have primed symbols. Conversely, any set of $2k$ right lines on the surface, such that an odd number have primed symbols, determines a unique set of $2k$ C^{n-2} which do not intersect. The number of such sets is thus 2^{2k-1}. A given C^{n-2} belongs to $2k$ sets. Two given skew C^{n-2} both belong to two sets. Three skew C^{n-2} define a unique set.

Let S denote a set of the given type. Each curve of the set determines a unique C^{n-2} which intersects it in $k - 1$ points and which is seen at once from its symbol to intersect each of the other curves of S in $k - 2$ points. The $2k$ C^{n-2} determined in this way by the C^{n-2} of S are skew to each other. Hence they form a set S' of the given type. It follows at once from the method of formation that the correspondence between S and S' is involutorial.

We have supposed $n > 3$, $k > 2$. In case $k = 2$, there exist, in addition to the above sets of skew curves, eight sets of four skew C^{n-2} such that all the C^{n-2} of a set intersect a fixed line on the surface. Each line on the surface determines a set and the sets can be arranged in conjugate pairs.

In case $n = 3$, the C^{n-2} are right lines and the above configurations are components, merely, of known configurations of right lines on rational surfaces which have their plane sections of genus unity.

A similar theory can be developed when the number of simple fundamental points is odd, $\Delta = 2k + 1$, so that the surface is of odd order, but in this case the correspondence between the sets, as defined above, is not one to one.

URBANA, ILL.,
February 21, 1916.

ON SEPARATED SETS.

BY PROFESSOR W. A. WILSON.

(Read before the American Mathematical Society, April 29, 1916.)

In the March number of the BULLETIN appeared a discussion of the definition of Lebesgue integrals given in Pierpont's Theory of Functions of Real Variables, volume II, by Fréchet and the author. The questions there discussed are much simplified if use is made of the *outer* and *inner associated sets* of a point set, concepts due, I believe, to W. H. Young.

These sets are defined in the text mentioned, but for the sake of convenience I shall give their definitions here. They arise at once from the definitions of upper and lower measure of a point set. Let A be the set under consideration. Let it be enclosed in an enumerable set R of rectangular cells, of which the sum of the areas is finite and may be denoted by $\bar{R}$. The minimum of all the possible values of $\bar{R}$ is called the upper measure of A and denoted by $\overline{\text{Meas}}\ A$. Now if a sequence $\{R_n\}$ of the rectangular sets is so chosen that $\lim_{n=\infty} \bar{R}_n = \overline{\text{Meas}}\ A$, their divisor (or set of points common to all of them) will contain A, and will be a measurable point set of measure equal to $\overline{\text{Meas}}\ A$ by the ordinary laws of measurable sets. Such a set is called an outer associated set of A and may be denoted by A_e. There will be an infinity of such sets corresponding to a single A, but for each A_e, $\text{Meas}\ A_e = \overline{\text{Meas}}\ A$ and the sets A_e differ only by a set of measure zero. The inner associated sets are defined as follows. Let A be enclosed in a rectangular cell Q, let $B = Q - A$, and let B_e be an outer associated set of B. Let $A_i = Q - B_e$. Then A_i is contained in A, is measurable and $\text{Meas}\ A_i = \text{Meas}\ Q - \text{Meas}\ B_e = \text{Meas}\ Q - \overline{\text{Meas}}\ B = \underline{\text{Meas}}\ A$, by the definition of lower measure. This set A_i is called an inner associated set of A. Young has also shown that any A_i may be regarded as the union of an enumerable set of complete sets C_n contained in A and such that $\lim_{n=\infty} \text{Meas}\ C_n = \underline{\text{Meas}}\ A$. The importance of these sets is obvious;

their existence makes it possible in questions concerning the upper and lower measures of any set A to replace A by a *measurable* set containing A and of measure equal to $\overline{\text{Meas}}\, A$ or contained in A and of measure equal to $\underline{\text{Meas}}\, A$.

Applying these notions to separated sets, we have at once the result that, *if A and B are separated sets, there exist measurable sets A_1 and B_1 enclosing A and B respectively and such that Meas $A_1 = \overline{\text{Meas}}\, A$ and Meas $B_1 = \overline{\text{Meas}}\, B$; and further that the measure of the divisor of any such pair, A_1 and B_1, is zero.* The first part of the theorem is obvious; we may take A_e for A_1 and B_e for B_1. To prove the second part, let A_2 and B_2 be measurable sets enclosing A and B respectively and such that the measure of their divisor is zero according to the definition of separated sets. Let A_3 be the divisor of A_1 and A_2, and A_4 the remainder of A_1. Now A_3 contains A, since both A_1 and A_2 do. Therefore Meas $A_3 \geqq \overline{\text{Meas}}\, A =$ Meas A_1. Since A_3 is also $\leqq A_1$, Meas $A_3 =$ Meas A_1 and Meas $A_4 = 0$. Similar results hold for B_3 and B_4, defined in like manner. Thus the divisor of A_1 and B_1 is contained in the divisor of A_2 and B_2, save for at most a set of zero measure made up from A_4 and B_4. Therefore the measure of the divisor of A_1 and B_1 is zero.

The theorem questioned by Fréchet is the following: *Let A and B be separated sets and C their union; then $\overline{\text{Meas}}\, C = \overline{\text{Meas}}\, A + \overline{\text{Meas}}\, B$.* The proof can now be given without the use of the ϵ_n-enclosures, which seem to have caused all the trouble. Let A_e, B_e, and C_e be outer associated sets of A, B, and C respectively. Let A_1 be the divisor of A_e and C_e; let B_1 be the divisor of B_e and C_e; and let C_1 be the union of A_1 and B_1. Also let D be the divisor of A_1 and B_1. Then $A \leqq A_1 \leqq A_e$, $B \leqq B_1 \leqq B_e$ and $C \leqq C_1 \leqq C_e$; hence Meas $A_1 = \overline{\text{Meas}}\, A$, Meas $B_1 = \overline{\text{Meas}}\, B$, Meas $C_1 = \overline{\text{Meas}}\, C$ and by the previous paragraph Meas $D = 0$. Therefore

$$\begin{aligned}\overline{\text{Meas}}\, C &= \text{Meas}\, C_1\\ &= \text{Meas}\, A_1 + \text{Meas}\, B_1 - \text{Meas}\, D\\ &= \overline{\text{Meas}}\, A + \overline{\text{Meas}}\, B - 0,\end{aligned}$$

which was to be proved.

We can also go farther and say that *if the set C is the union*

of A and B, and $\overline{\text{Meas}}\, C = \overline{\text{Meas}}\, A + \overline{\text{Meas}}\, B$, the sets A and B are separated. Proceeding as in the previous theorem,

$$\begin{aligned}\overline{\text{Meas}}\, C &= \text{Meas}\, C_1\\ &= \text{Meas}\, A_1 + \text{Meas}\, B_1 - \text{Meas}\, D\\ &= \overline{\text{Meas}}\, A + \overline{\text{Meas}}\, B - \text{Meas}\, D.\end{aligned}$$

But since $\overline{\text{Meas}}\, C = \overline{\text{Meas}}\, A + \overline{\text{Meas}}\, B$, Meas $D = 0$. Thus we have A and B enclosed in measurable sets A_1 and B_1 respectively, of which the divisor D has measure zero. This is the requirement for separated sets.

Regarding the example used by Fréchet and questioned by Pierpont, it can be shown that *any separated partition of a measurable set A will be made up of measurable sets only.*

It is sufficient to prove this for the case that the partition consists of two sets only. For, let B be any one of an enumerable set of separated sets making up A and let U be the union of the remainder. It is readily seen from the definition of separated sets that B and U are separated. Hence it is sufficient to prove B and U measurable.

To do this let B_e and U_e be outer associated sets of B and U respectively, and let B_1 be the divisor of A and B_e, and U_1 the divisor of A and U_e. Then by previous results B_1 and U_1 are measurable, Meas $B_1 = \overline{\text{Meas}}\, B$ and Meas $U_1 = \overline{\text{Meas}}\, U$, and the measure of their divisor is zero.

As A is the union of B_1 and U_1, the set B_1 consists of B and certain points of U contained in B_1. But the divisor of B_1 and U_1 is a null set, hence the divisor of B_1 and U is a null set. Therefore those points of B_1 not belonging to B have measure zero and thus B is the difference between the measurable set B_1 and a null set. Hence B is measurable. In like manner U is measurable. But B was any set of those making up A_1 and so the theorem is proved.

This with the previous theorem gives the important result that no measurable set can be made up of an enumerable set of non-measurable point sets and have the additive property, *i. e.*, $\overline{\text{Meas}}\, A = \overline{\text{Meas}}\, A_1 + \overline{\text{Meas}}\, A_2 + \cdots$, preserved.

New Haven, Conn.

SINGULAR POINTS OF TRANSFORMATIONS AND TWO-PARAMETER FAMILIES OF CURVES.

BY DR. W. V. LOVITT.

1. *Introduction.*

IN the *Transactions* for October, 1915, I discussed some singularities of a point transformation in three variables

$$(1) \qquad x = \phi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w).$$

Let a particular one of the singular points in question be denoted by P, and let S denote the surface through P in the uvw-space defined by setting the jacobian of the transformation equal to zero. The point P and the surface S are transformed by (1) into a point P_1 and surface S_1 in the xyz-space.

In the present paper there is found on the surface $S_1(x, y, z)$ a curve (d_1) which is the envelope of a one-parameter family of curves properly chosen from the two-parameter family (1). We find in the uvw-space that plane of directions which transforms into the direction of the curve (d_1) in the xyz-space.

2. *Initial Assumptions.*

Let us consider a real point transformation of three-space

$$(1) \qquad x = \phi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w)$$

with determinant

$$J(u, v, w) \equiv \begin{vmatrix} \phi_u & \phi_v & \phi_w \\ \psi_u & \psi_v & \psi_w \\ \chi_u & \chi_v & \chi_w \end{vmatrix}.$$

The functions ϕ, ψ, χ are not necessarily analytic but it will be presupposed that

(*a*) the functions ϕ, ψ, χ are of class C'''* in a neighborhood of the origin $(u, v, w) = (0, 0, 0)$;

* We shall say that a single-valued function f of u, v, w is of class C''' if $f(u, v, w)$ and its partial derivatives of orders one, two, and three are continuous in a region in which f is defined.

(*b*) the following initial conditions are satisfied:

$$\phi(0, 0, 0) = \psi(0, 0, 0) = \chi(0, 0, 0) = 0;$$

(*c*) $J(0, 0, 0) = 0$;

(*d*) at the origin $(u, v, w) = (0, 0, 0)$ at least one of the determinants of the matrix

$$\begin{Vmatrix} J_u & J_v & J_w \\ \phi_u & \phi_v & \phi_w \\ \psi_u & \psi_v & \psi_w \\ \chi_u & \chi_v & \chi_w \end{Vmatrix} \tag{2}$$

is different from zero.

There is no loss of generality in assuming, as indicated in the conditions, that the singular point P is at the origin in the uvw-space, and that the transform of P by (1) is the origin P_1 in the xyz-space. Neither will generality be lost if we assume for convenience that the determinant

$$H_1 \equiv \begin{vmatrix} J_u & J_v & J_w \\ \psi_u & \psi_v & \psi_w \\ \chi_u & \chi_v & \chi_w \end{vmatrix} \tag{3}$$

is that one of the matrix (2) which does not vanish at the origin.

By our assumptions (*b*) and (*c*) the equations

$$J(u, v, w) = 0, \quad y = \psi(u, v, w), \quad z = \chi(u, v, w) \tag{4}$$

have the initial solution $(u, v, w, y, z) = (0, 0, 0, 0, 0)$. The hypothesis (*d*) justifies the assumption that the determinant (3) is different from zero, as we have seen. Hence by the usual theorems of implicit functions there exists a neighborhood $(0, 0, 0, 0, 0)_\epsilon$* in which no two solutions (u, v, w, y, z) of equations (4) have the same projection (y, z), and a neighborhood $(0, 0)_\delta$ of the point $(y, z) = (0, 0)$ in which equations (4) determine u, v, w as functions of class C'' of y and z,

$$u = u(y, z), \quad v = v(y, z), \quad w = w(y, z) \tag{5}$$

* For these theorems see Bliss, Princeton Colloquium Lectures, pp. 8–9. By the notation $(0, 0, 0, 0, 0)_\epsilon$ is meant a neighborhood

$$|u| < \epsilon, \quad |v| < \epsilon, \quad |w| < \epsilon, \quad |y| < \epsilon, \quad |z| < \epsilon$$

of the point $(0, 0, 0, 0, 0)$.

defining values (u, v, w, y, z) in the neighborhood $(0, 0, 0, 0, 0)_\epsilon$. By substituting these results in the third of equations (1), a surface

$$(S_1) \qquad x = X(y, z)$$

is found, which is the transform by (1) of the surface S.

3. *The Envelope Curve d_1.*

We now interpret equations (1) as a two-parameter family of curves with the parameters v, w. Under the assumption (d), the surface S_1 is the envelope of the curves (1).* If a one-parameter family of curves be chosen from the set (1), this family will not in general have an enveloping curve. The condition that a curve

$$(d_1) \qquad x = x(\alpha) = X[y(\alpha), z(\alpha)], \quad y = y(\alpha), \quad z = z(\alpha)$$

on the surface S_1 shall be an envelope may be derived as follows.†

If we substitute $y(\alpha)$, $z(\alpha)$ in the functions $v(y, z)$, $w(y, z)$ defined by equations (5) two functions $v(\alpha)$, $w(\alpha)$ are determined and a one-parameter family of curves is defined when $v(\alpha)$, $w(\alpha)$ are substituted in (1). These curves are tangent to the curve (d_1) if y and z are determined as functions of α so that

$$x_\alpha = \phi_y y_\alpha + \phi_z z_\alpha = m\phi_u, \quad y_\alpha = m\psi_u, \quad z_\alpha = m\chi_u,$$

u, v, and w being thought of as functions of y and z. The three determinants of the matrix

$$\left\| \begin{matrix} \phi_y y_\alpha + \phi_z z_\alpha & y_\alpha & z_\alpha \\ \phi_u & \psi_u & \chi_u \end{matrix} \right\|$$

must therefore be zero, *i. e.*, the three equations

$$(\phi_u - \phi_y \psi_u) y_\alpha - \psi_u \phi_z z_\alpha = 0$$

$$\chi_u \phi_y y_\alpha - (\phi_u - \chi_u \phi_z) z_\alpha = 0,$$

$$\chi_u y_\alpha - \psi_u z_\alpha = 0,$$

---

* W. V. Lovitt, "A type of singular points for a transformation of three variables," *Transactions*, vol. 16 (1915), p. 377.

† Mason-Bliss, "The properties of curves in space which minimize a definite integral," *Transactions*, vol. 9 (1908), pp. 440–466.

must be satisfied. The coefficients of y_α, z_α in these equations cannot all vanish since at least one of the derivatives ϕ_u, ψ_u, χ_u is different from zero at the point P_1. That any two of the equations are a consequence of the third may be shown by expanding the determinant of any pair of the equations and using the relation

$$\phi_u - x_y\psi_u - x_z\chi_u = 0.$$

The determination of a one-parameter family of curves having an enveloping curve (d_1) is therefore to be effected by solving one of the above equations. It has the form

$$A(y, z)y_\alpha + B(y, z)z_\alpha = 0,$$

when u, v, and w are replaced by their values in terms of y and z from equations (5). Since this differential equation is of the first order there exists one and only one integral curve

$$y = y(\alpha), \quad z = z(\alpha)$$

in the yz-plane, passing through the point $y = z = 0$ for $\alpha = 0$. The equations of the family of curves tangent to (d_1) are found by substituting $y(\alpha)$, $z(\alpha)$ in the expressions for v and w in terms of y and z from equations (5) and then putting the resulting functions $v(\alpha)$, $w(\alpha)$ in equations (1). A family of extremals

$$x = \phi(u, \alpha), \quad y = \psi(u, \alpha), \quad z = \chi(u, \alpha)$$

is thus found, which are tangent to (d_1) when $u = u(\alpha)$. The equation of the envelope (d_1) will then be

$$x = \phi[u(\alpha), \alpha], \quad y = \psi[u(\alpha), \alpha], \quad z = \chi[u(\alpha), \alpha].$$

We have then the following theorem:

THEOREM 1: *Given a family of curves*

$$x = \phi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w),$$

if on a particular curve C_1 the determinant J vanishes at the point P_1 and one at least of the determinants of the matrix (2) *is different from zero at P_1, then the family of curves has an enveloping surface S_1 which touches C_1 at P_1 and for which P_1 is not a singular point. On the surface S_1 there exists a unique curve (d_1) without singular points, which passes through the point P_1 and envelopes a one-parameter family of curves, con-*

taining the curve C_1, *which are the transforms by* (1) *of the lines, in the uvw-space, parallel to the u-axis.*

4. *The Plane of Directions which Transforms into the Direction of* d_1.

Urner* has shown that the necessary and sufficient condition that two non-tangent curves through P which have their directions distinct from the critical direction be rendered tangent by the transformation, is that the plane of their tangents at the point contain the line having the critical direction. Furthermore each plane of directions through the critical direction is compressed into a single direction. We ask, what is the plane of directions which is compressed into the direction of the envelope (d_1) at P_1?

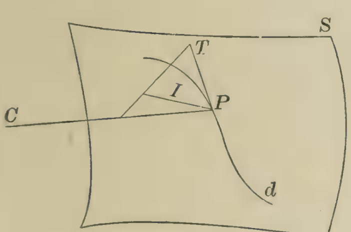
This plane must contain the line of critical direction

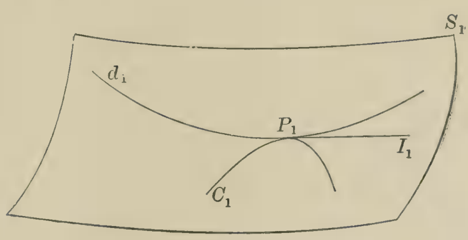
$$(I) \qquad \frac{u}{I_1} = \frac{v}{I_2} = \frac{w}{I_3}.$$

It must contain the line

$$(C) \qquad v = w = 0.$$

Thus the plane is completely determined, unless these directions coincide; but this cannot, in general, happen.





Designate by λ the plane determined by C and I. Every direction in λ except I goes into the same direction I_1 in the xyz-space. In particular the direction T which is the intersection of λ with the tangent plane to the surface S at P goes into the direction I_1. To the curve (d_1) on the surface S_1 there corresponds a unique curve (d) on the surface S and from

* S. E. Urner, "Certain singularities of point transformations in space of three dimensions," *Transactions*, vol. 13 (1912), pp. 232–264.

what we have just shown the curve (d) must be tangent to the plane λ at the point P.

We determine the differential equation for the curve (d) as follows: obtain the equation of the plane λ. It is

$$I_3 v - I_2 w = 0.$$

The line T is the intersection of this plane with the tangent plane to the surface S at the point P. The projection of its direction on the vw-plane is

$$I_2 dw - I_3 dv = 0.$$

The integral curve of this equation which passes through the point P is the projection of the curve (d) upon the vw-plane. This integral together with the surface S completely determines the curve (d).

We have then the following theorem:

THEOREM 2. *The plane of directions in the uvw-space determined by the critical direction I $(I_1 : I_2 : I_3)$ and the line $v = w = 0$ and containing the tangent line T to the curve (d) transforms into a single direction I_1 in the xyz-space. The direction I_1 is the direction of the tangent to the curve (d_1), which curve is the transform of the curve (d) by means of equations* (1).

5. *Illustration.*

The transformation

$$x = u^2, \quad y = u + v, \quad z = v + w$$

has for its jacobian $J \equiv 2u$. Thus the jacobian surface S is the surface $u = 0$. Substitution in equation (3) gives us $H_1 \equiv 2 \neq 0$. The critical direction is given by $1 : 1 : 1$. The line (d) on S and through P $(0, v_0, w_0)$ which transforms into (d_1) on S_1 is the line

$$(d) \qquad v + w = v_0 + w_0, \quad u = 0.$$

The transform of (d) is the line

$$(d_1) \qquad z = v_0 + w_0, \quad x = 0.$$

The surface S_1 is given by $x = 0$. The lines

$$v = c_1, \quad w = c_2, \quad u = t,$$

which are parallel to the u-axis, are transformed into

$$x = (y - c_1)^2, \quad z = c_1 + c_2,$$

which are seen to be tangent to the surface S_1. It is evident from the last equations that those one-parameter families of parabolas which lie in the planes parallel to the xy-plane have envelopes, and that no others have. These envelopes are the curves (d_1).

PURDUE UNIVERSITY.

AN ELEMENTARY BOUNDARY VALUE PROBLEM.

BY PROFESSOR DUNHAM JACKSON.

(Read before the American Mathematical Society, April 29, 1916.)

IT is intuitively obvious that if a simple continuous curve is given in the (x, y)-plane, and a continuous distribution of values along the curve, there will exist functions of x and y which are continuous in both variables together, and which take on the prescribed values along the curve. It is the purpose of the present note to give an analytic proof of this fact, by elementary means, and, in particular, without reference to potential theory.* The problem will be treated first for the case of a rectifiable curve, then for an arbitrary Jordan curve.

Let the equations

$$x = f(s), \quad y = \varphi(s), \qquad (0 \leqq s \leqq l),$$

define a simple closed rectifiable curve C, the variable s standing for the length of arc, and l for the total length of the curve. It is assumed that the functions $f(s)$ and $\varphi(s)$ are continuous throughout their interval of definition, and that $f(0) = f(l)$, $\varphi(0) = \varphi(l)$, but that with this exception no one pair of values (x, y) is given by two distinct values of s. Let $F(s)$ be an arbitrary continuous function defined throughout the same interval, subject to the condition that $F(0) = F(l)$.

* I understand that Mr. R. E. Gleason has had occasion to deal with a similar problem in connection with a paper recently presented to the Society; see BULLETIN, vol. 22 (1916), pp. 278–279.

We shall consider the function $J(x, y) = J_1/J_0$, where

$$J_1(x, y) = \int_0^l \frac{F(s)}{\rho^2} ds, \quad J_0(x, y) = \int_0^l \frac{ds}{\rho^2},$$

and

$$\rho = \sqrt{[x - f(s)]^2 + [y - \varphi(s)]^2}.$$

The function J is obviously defined and continuous throughout the (x, y)-plane, with the exception of the points of the curve C. We shall show that if the point (x, y) approaches a point P_0 of C, with the coordinates $(f(s_0), \varphi(s_0))$, the value of $J(x, y)$ will approach $F(s_0)$ as a limit.* There is clearly no loss of generality in assuming that s_0 is distinct from 0 and l.

We have to deal with the difference

$$J(x, y) - F(s_0) = \frac{1}{J_0} \int_0^l [F(s) - F(s_0)] \frac{ds}{\rho^2}.$$

Let ϵ be any positive quantity. Let $\delta > 0$ be chosen so that $|F(s) - F(s_0)| \leqq \frac{1}{2}\epsilon$ for $|s - s_0| \leqq \delta$. Then

$$\left| \int_{s_0-\delta}^{s_0+\delta} [F(s) - F(s_0)] \frac{ds}{\rho^2} \right| \leqq \tfrac{1}{2}\epsilon J_0,$$

regardless of the position of the point (x, y), provided only that it does not lie on C. Let γ be the minimum distance from P_0 to a point of C for which $|s - s_0| \geqq \delta$. This minimum will be positive, since the distance is a continuous function of s which does not reduce to zero. Denoting by ρ_0 the distance

$$\rho_0 = \sqrt{[x - f(s_0)]^2 + [y - \varphi(s_0)]^2},$$

a quantity which depends on x and y but not on s, we can be sure that if $\rho_0 \leqq \frac{1}{2}\gamma$, then $\rho \geqq \frac{1}{2}\gamma$ for $|s - s_0| \geqq \delta$. Consequently, if M is the maximum of $|F(s)|$,

$$\left| \int_0^{s_0-\delta} [F(s) - F(s_0)] \frac{ds}{\rho^2} \right| \leqq \int_0^{s_0-\delta} 2M \cdot \frac{4ds}{\gamma^2} = \frac{8M(s_0 - \delta)}{\gamma^2};$$

* It is our purpose merely to indicate a single solution of the boundary value problem; when on solution is given, it is of course possible immediately to find infinitely many others.

and similarly

$$\left| \int_{s_0+\delta}^{l} [F(s) - F(s_0)] \frac{ds}{\rho^2} \right| \leqq \frac{8M(l - s_0 - \delta)}{\gamma^2}.$$

We see accordingly that

$$(1) \qquad | J(x, y) - F(s_0) | \leqq \tfrac{1}{2}\epsilon + \frac{8Ml}{\gamma^2 J_0},$$

provided that (x, y) is a point at a distance from P_0 not greater than $\frac{1}{2}\gamma$, and not lying on C. It remains only to show that as (x, y) approaches P_0, the value of J_0 becomes infinite. For if this is established, the second term on the right-hand side of (1) will be less than $\frac{1}{2}\epsilon$ when (x, y) is sufficiently near to P_0, and it will follow that

$$| J(x, y) - F(s_0) | < \epsilon.$$

Let η be an arbitrarily small positive quantity. If $\rho_0 \leqq \frac{1}{2}\eta$ and $| s - s_0 | \leqq \frac{1}{2}\eta$, it is certain that $\rho \leqq \eta$, since it has been assumed that s represents the length of arc along the curve. For all points within a distance $\frac{1}{2}\eta$ of P_0, therefore,

$$J_0 > \int_{s_0-\frac{1}{2}\eta}^{s_0+\frac{1}{2}\eta} \frac{ds}{\rho^2} \geqq \int_{s_0-\frac{1}{2}\eta}^{s_0+\frac{1}{2}\eta} \frac{ds}{\eta^2} = \frac{1}{\eta},$$

which can be made arbitrarily large by taking η sufficiently small. This completes the proof.

Now let C be an arbitrary closed Jordan curve, given by a pair of equations

$$x = f(t), \quad y = \varphi(t), \qquad 0 \leqq t \leqq a,$$

where the functions f and φ are continuous, and do not yield any one point twice, except for $t = 0$ and $t = a$. It will be convenient to think of f and φ as defined for all real values of t, with the period a; they will be continuous without exception. Let $F(t)$ be an arbitrary continuous function of period a.

Let $\omega_1(\eta)$ and $\omega_2(\eta)$ be the maxima of $| f(t'') - f(t') |$ and $| \varphi(t'') - \varphi(t') |$ respectively for $| t'' - t' | \leqq \eta$, and let $\omega(\eta) = \omega_1(\eta) + \omega_2(\eta)$. Then $\omega(\eta)$, defined for $\eta \geqq 0$, is a function which is positive or zero, and never decreases when η increases. Furthermore, $\lim_{\eta=0} \omega(\eta) = 0$, because of the uniform continuity of f and φ; and, more generally, $\omega(\eta)$ is continuous for all positive values of η, since $\omega(\eta' + \eta'')$

$\leqq \omega(\eta') + \omega(\eta'')$. Let $\beta = \chi(\eta) = \omega(\eta) + \eta$. This new function is continuous, reduces to zero for $\eta = 0$, and *always increases* when η increases. To each value of $\beta \geqq 0$ corresponds one and just one value of η; the inverse function

$$\eta = \psi(\beta)$$

is itself increasing and continuous, and, in particular, $\lim_{\beta=0} \psi(\beta) = 0$.

We are ready now to write down a solution of the boundary value problem. We shall set $J(x, y) = J_1/J_0$, where

$$J_1(x, y) = \int_0^a \frac{F(t)dt}{[\psi(\frac{1}{2}\rho)]^2}, \quad J_0(x, y) = \int_0^a \frac{dt}{[\psi(\frac{1}{2}\rho)]^2},$$

and

$$\rho = \sqrt{[x - f(t)]^2 + [y - \varphi(t)]^2}.$$

It is true again that $J(x, y)$ is defined and continuous at all points (x, y) not lying on the given curve. Guided by the earlier demonstration, we shall begin the proof that $J(x, y)$ is a function having the desired property, by showing that J_0 becomes infinite as (x, y) approaches a point P_0: $(f(t_0), \varphi(t_0))$ of the curve.

Let β be an arbitrarily small positive quantity, and $\eta = \psi(\beta)$. If $| t - t_0 | \leqq \eta$, then

$$| f(t) - f(t_0) | \leqq \omega_1(\eta), \quad | \varphi(t) - \varphi(t_0) | \leqq \omega_2(\eta),$$

and hence the distance from P_0 to the point $(f(t), \varphi(t))$ is subject to the inequality

$$\sqrt{[f(t) - f(t_0)]^2 + [\varphi(t) - \varphi(t_0)]^2} \leqq \omega(\eta) \leqq \chi(\eta) = \beta.$$

Denoting by ρ_0 the distance from P_0 to the point (x, y), we see that when

$$\rho_0 < \beta \tag{2}$$

we can be sure that $\rho < 2\beta$ for values of t in the interval just named, and hence

$$\tfrac{1}{2}\rho \leqq \beta, \quad \psi(\tfrac{1}{2}\rho) \leqq \psi(\beta) = \eta.$$

It follows that if (x, y) is within the neighborhood of P_0 defined by the inequality (2),

$$J_0(x, y) > \int_{t_0-\eta}^{t_0+\eta} \frac{dt}{[\psi(\frac{1}{2}\rho)]^2} \geqq \int_{t_0-\eta}^{t_0+\eta} \frac{dt}{\eta^2} = \frac{2}{\eta} = \frac{2}{\psi(\beta)}.$$

As $\psi(\beta)$ approaches zero with β, the assertion with regard to J_0 is justified.

Returning to the consideration of the numerator J_1 and the quotient J, let ϵ be an arbitrarily small positive quantity, and δ a positive quantity such that $|F(t) - F(t_0)| \leqq \frac{1}{2}\epsilon$ for $|t - t_0| \leqq \delta$; let γ be the minimum distance from P_0 to a point of C for which $|t - t_0| \geqq \delta$, a distance which is surely positive,* and finally let M be the maximum value of $|F(t)|$. If $\rho_0 \leqq \frac{1}{2}\gamma$, it is certain that $\rho \geqq \frac{1}{4}\gamma$ for $|t - t_0| \geqq \delta$. We see that the following inequalities hold:

$$\left|\int_{t_0-\delta}^{t_0+\delta} \frac{F(t) - F(t_0)}{[\psi(\frac{1}{2}\rho)]^2}\, dt\right| \leqq \frac{\epsilon}{2}\int_{t_0-\delta}^{t_0+\delta} \frac{dt}{[\psi(\frac{1}{2}\rho)]^2} < \tfrac{1}{2}\epsilon J_0;$$

$$\left|\int_{0}^{t_0-\delta} \frac{F(t) - F(t_0)}{[\psi(\frac{1}{2}\rho)]^2}\, dt\right| \leqq \int_{0}^{t_0-\delta} \frac{2M dt}{[\psi(\frac{1}{4}\gamma)]^2} = \frac{2M}{[\psi(\frac{1}{4}\gamma)]^2}(t_0 - \delta);$$

$$\left|\int_{t_0+\delta}^{a} \frac{F(t) - F(t_0)}{[\psi(\frac{1}{2}\rho)]^2}\, dt\right| \leqq \frac{2M}{[\psi(\frac{1}{4}\gamma)]^2}(a - t_0 - \delta);$$

and by combination of these,

$$|J(x, y) - F(t_0)| = \left|\frac{1}{J_0}\int_{0}^{a} \frac{F(t) - F(t_0)}{[\psi(\frac{1}{2}\rho)]^2}\, dt\right| \leqq \frac{\epsilon}{2} + \frac{2Ma}{[\psi(\frac{1}{4}\gamma)]^2 J_0}.$$

If (x, y) is sufficiently near to P_0, the value of J_0 will be so large that the second term on the right is less than $\frac{1}{2}\epsilon$, and we shall have the inequality which establishes the theorem to be proved,

$$|J(x, y) - F(t_0)| < \epsilon.$$

It may be remarked that similar reasoning can be applied to Jordan curves that are not closed, or to a system of any finite number of Jordan curves, no two of which have a point in common.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

* We are assuming here that t is restricted to the interval $0 \leqq t \leqq a$, and are making for convenience the further assumption, of no essential significance, that t_0 is an interior point of the same interval.

CONCERNING REVIEWS.

On reading certain book reviews that have appeared in recent numbers of the Bulletin, one is reminded of Addison's complaint that rather than to dwell upon the excellencies of a work some reviewers imagine they have discharged their duty when they have succeeded in pointing out slight faults and errors, forgetting that

> "Errors, like straws, upon the surface flow;
> He who would search for pearls must dive below."

As an instance justifying this complaint I wish to cite the review of the "Memorabilia Mathematica" in the January number of the Bulletin. Except for an extract from the preface of the book and its table of contents, which is erroneously quoted (there are twenty-one chapter headings instead of the seventeen quoted by the reviewer, and two of those are wrongly quoted), one seeks in vain for a word that would enlighten the reader as to the contents of the book. The remainder of the review, as far as it deals with the work under consideration, is limited to trivial errors and petty fault-finding. Of what possible interest can it be to the reader to be told that in one place T' should be replaced by T', or that "inapt" appears where the original has the misspelled form "unapt," or again, that Newton's utterance "I don't know what I may seem to the world" as quoted by Parton is given by Brewster in the form "I do not know what I may appear to the world"? Surely an author might deem himself fortunate whose work were blemished by no greater faults!

Suppose the reviewer's five-page review were reviewed according to his own standard. He writes "Porton" for "Parton," "Euclyde" for the "Euclide" of the original, and "hut" for "hyt" in the line "Yn Egypte he tawghte hyt ful wyde." His quotation from Prior

> "Circles to square, and cubes to double,
> Would give a man excessive trouble;"

should read

> "Circles to square, and Cubes to double,
> Would give a Man excessive Trouble:"*

* Matthew Prior, Cambridge English Classics, Cambridge (1905), p. 248.

The original form of the line

> "God said, 'Let Newton be!' and all was light,"

from Pope's Epitaph intended for Sir Isaac Newton is not

> "God said, *Let Newton be*! and all was light,"

as the reviewer has it, but

> "GOD said, *Let Newton be*! and all was Light."*

There are other errors of a like character but enough have been cited to lend support to Addison's dictum that there never was a critic who made it his business to lash the faults of others who was not guilty of greater faults himself.

In conclusion I must call attention to one or two more serious errors in the review in question. On page 188 we are told that in Ahrens's Scherz und Ernst in der Mathematik names of living mathematicians are rarely met with. The volume in question contains by actual count 20 quotations from Klein, 18 from Poincaré, 10 from M. Cantor, 7 each from Hilbert and Frobenius, 6 from G. Cantor, and so on through more than a score of names of men either now living or deceased since the book appeared in 1904. Again, the Memorabilia Mathematica contains 1,140 quotations instead of some 1,200 as stated by the reviewer. The seven-line reference on page 190 to the reviewer's own paper is irrelevant to the matter in hand. A most curious slip occurs on page 191 in the reviewer's observation "for Reid, M." read "Reid, T." the line criticized being, "Reid, M. as an exercise in language."

ROBERT E. MORITZ.

THE UNIVERSITY OF WASHINGTON.

WITH regard to the collection of quotations which Professor Moritz edits, the reviewer does not find that he has made a single statement which may be legitimately termed inaccurate, or which is liable to give a wrong impression as to the merits of the editor's redaction—a redaction which the editor himself appears to regard as containing "pearls" unnoticed by the reviewer. Let us see what his strictures amount to.

---

* Warburton's "The Works of Alexander Pope, as they were delivered to the Editor a little while before his death, etc." London (1760), vol. 6, p. 99.

In spite of the authority of the great Oxford Dictionary he contends that "some 1200" may not be used to refer to 1140! Again, the reviewer is accused of error in tabulation of the contents. These consist, presumably, (1) in stating that the quotations were classed under 20 headings when the editor claims 21 ("Persons and anecdotes, A–M," "Persons and anecdotes, N–Z," the reviewer was guilty of combining under one heading: "Persons and anecdotes"); (2) in asserting that "Definition and object of mathematics" was a heading when the first word should have been in the plural form; and (3) in leaving it to the reader to infer that two of the 20 headings, "Mathematics as a Fine Art" and "Mathematics as a Language," were indicated by "Mathematics as a fine art, as a language"; and similarly that three headings were indicated by "Mathematics and logic, and philosophy, and science."

In response to the inquiries of Professor Moritz the following replies may be vouchsafed: (*a*) In the Thomson and Tait quotation T' has no meaning, while the calculus notation T' is peculiarly pregnant with suggestion; (*b*) it is not true that unapt is misspelled for inapt in Orr's mnemonic—this may be verified by the simple expedient of consulting the Century Dictionary; (*c*) it may be learned from any first-class librarian that Parton is a worthless authority in connection with any statement concerning Newton.

Let us now take up five examples, somewhat different in character, to illustrate Professor Moritz's methods of criticism.

1. In his book he gives two of the seven lines written by Pope as an epitaph on Sir Isaac Newton and refers to those two lines as the epitaph in question; the reviewer remarked that this statement was inaccurate and gave the full quotation, with a reference to Elwin and Courthope's standard edition of Pope's works. In this quotation there is not the slightest misprint in capitalization, in italics, or in punctuation. That some other edition, no more authoritative, has a different capitalization in one line is entirely irrelevant.

2. The same method is applied to the quotation by Prior, again given with exact reference by the reviewer. There is not a particle of variation between the original and that indicated in the review.

3. Again, the reviewer wrote: "on page 405 of the index for Reid, M. read Reid, T." To be more explicit, 8 quotations are attributed to Thomas Reid in the Memorabilia. These

are all incorrectly listed in the index on page 405, under Reid, M. Instead, therefore, of a "most curious slip" on the part of the reviewer, yet another has been made by Professor Moritz himself. But his slips in this sentence are not confined to one, or even two. Without any foundation whatever the reviewer is accused of criticizing a line "Reid, M. as an exercise in language." This line occurs nowhere in the book. True one does find "Reidt, M. as an exercise in language," to which the reviewer made no reference, but here he now finds another slip, for instead of M. should be F.*

4. The reviewer repudiates Professor Moritz's statement of what he wrote concerning Ahrens's work. What he *did* write was as follows: "A 24-page detailed index of subjects and authors provides the means for rapid orientation. Names of living mathematicians are rarely met with, but references to the "old masters" such as Abel, Euclid, Gauss, Helmholtz, Lagrange, Laplace, Steiner and Weierstrass are very numerous." Even if "a score of names" of living mathematicians *may* be found in the 24-page index, the statement of the reviewer has not been shown to be in the smallest degree inaccurate.

5. With exact reference to Halliwell's Rara Mathematica the reviewer quoted some lines referring to Euclid. It is not true that "Euclide" should replace "Euclyde" in that quotation; it is true that "hyt" should replace "hut" in the third line, and the reviewer is glad to have his attention drawn to this slip in proof-reading.

The relation between the reviewer's and critic's statements thus set forth, render impotent the critic's remark concerning "other errors of a like character." In conclusion it may now be added that in his review the reviewer mentioned only a few of the three score of slips which he had noticed in the Memorabilia.

R. C. ARCHIBALD.

---

* On page 408 the biographer of Lord Kelvin is referred to as Sylvanus (instead of Silvanus) Thompson. The reviewer is indebted to Mr. W. J. Greenstreet for calling his attention to this same slip in his review.

SHORTER NOTICES.

Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi, with an introduction, critical notes, and an English version. By Louis Charles Karpinski. University of Michigan Studies, Humanistic Series, Vol. XI. New York, The Macmillan Company, 1915. viii + 164 pp. Price $2.

Of the various kinds of contributions to the history of mathematics, those which are based upon a study of original sources are, of course, the most important. It was through contributions of this type that Boncompagni's *Bullettino* exercised its great influence; it is through these studies that the Abhandlungen zur Geschichte der Mathematik have proved so valuable; and it is through its articles based upon original sources that the *Bibliotheca Mathematica* has made its reputation. In this critical study of original sources, however, our own scholars have thus far been very backward. This is entirely natural, because it is first necessary to study the secondary sources in any new branch, and it takes time to discover a problem and to find the opportunity for assisting in its solution. On this account those to whom the story of mathematics has a charm have thus far, in our country, been chiefly occupied in learning the literature and in delving into the pages of Cantor and his predecessors, or in studying as time allowed such works as those of Heath or Braunmühl.

It is for this reason that the appearance of this work by Professor Karpinski is particularly noteworthy. Many scholars publish the results of their studies in one line or another from time to time, and these results are often noteworthy; but to few is it given, as it has been in this case, to suggest a new excursion into one of the by-paths of the academic grove of a country. And yet this is what has been done in the work under review, for it stands as the first noteworthy original study of a European version of an early classic in mathematics.

It is well known that Al-Khowarizmi, early in the ninth century, wrote a work bearing the title "algebr w'almuqabala." It is also well known that this work was translated into Latin by Robertus Retinensis (Ketenensis, de Ketene, Ostiensis, Astensis, or Cestrensis), commonly known today as Robert

of Chester, one of the group of English scholars that revealed the most important scientific works of the Arabs to western Europe in the twelfth century. Here, however, the story comes to an end for most students. Occasionally some reader, browsing in the appendix to Libri's Histoire, comes across the Latin version ascribed to Gherardo of Cremona, and the Rosen translation (1831) from an Arabic manuscript in the Bodleian Library is accessible in any large scientific collection; but Robert of Chester's version has thus far remained beyond the reach of most students. Curiously enough, however, the work was at one time prepared for the printer, by Johann Scheybl, professor of mathematics at Tübingen, probably soon after 1550. For some reason Scheybl's text was never published, and this manuscript was not preserved, like various others of this writer, in the library of his university. After various wanderings it appeared in the stock of a German book dealer about fifteen years ago, and the writer of this review purchased it for a small sum, as an anonymous manuscript on mathematics, for the library of Columbia University. About a year later, upon examining it with some care, the name of Scheybl suggested that it might be one of his manuscripts, and photographs were made of other manuscripts in his hand at Tübingen. A comparison of the handwriting showed at once, and without question, that the Columbia manuscript was a lost one of Scheybl's. Moreover, it was seen that it contained a Latin version of Al-Khowarizmi which differed in various details from the one published by Libri and from the Arabic copy used by Rosen. It is this manuscript which Professor Karpinski has transcribed, translated, and annotated.

The form of the publication is very satisfactory. On the left-hand page is the Latin text; on the right-hand page is the translation; at the foot of each page are such notes as are necessary to show the variation of the Scheybl version from the Dresden and Vienna codices and from the Arabic manuscript used by Rosen. Professor Karpinski has wisely refrained from attempting a literal translation, since the transcribed text furnishes all needed material for the study of exact expressions; but he has given his readers that free type of translation which permits of a work being easily read and easily understood. All difficulties of any moment are removed by the extensive array of footnotes, and altogether there is little which one could desire that has not been given.

In one sense the Robert of Chester version is not as satisfactory as the one attributed to Gherardo of Cremona, but the latter had already been published by Libri, and hence it was desirable that the former should also be made available for study. In this connection it is also proper to mention another manuscript of Al-Khowarizmi's algebra, for a knowledge of which many students are indebted to Professor Karpinski. This version lay unnoticed in an Italian manuscript of the fifteenth century until the present reviewer happened upon it in the library of the well-known bibliophile George A. Plimpton, of New York, some years ago. This may possibly be the version of William of Luna, and some scholar should do for it what Professor Karpinski has done for the Robert of Chester translation.

Of the work of Al-Khowarizmi itself, this is not the place to speak, since we are concerned with the translation rather than the original. One problem concerning it may, however, be mentioned, namely, that which relates to the source of Al-Khowarizmi's knowledge. Essentially, the treatment is Greek, but no direct connection exists between it and such classical works as those of Euclid and Diophantus. Neither the problems nor the identical methods can be traced to any other source, although Al-Khowarizmi knew something of Hindu mathematics and the Greek authors were already becoming known in Bagdad where he was at work. The problem, therefore, is to determine whether any of this material is to be found in the works of minor Greek or Hindu writers, or possibly in the works of Persian and Chinese authors whose treatises have still to be critically examined.

The work closes with a carefully selected Latin glossary which will be helpful to students of mathematics of the medieval and renaissance periods. It is probably too much to expect that in any university series of this kind the reader is to be assisted by an index. Whether this is because of tradition or because of the economy of our universities it is hard to say; but we may be sure that on this occasion it is not due to the wishes of a student like Professor Karpinski. No one who, like the writer, has had occasion to refer to this work several times, and wishes to find such an item as Gherardo's supposed translation, can fail to regret the omission of this feature.

On the whole, it may be said that the work under review is

a very noteworthy contribution to the study of sources in the history of mathematics.

DAVID EUGENE SMITH.

A First School Calculus. By R. WYKE BAYLISS. London, Longmans, Green and Company, 1915. xii + 288 pp.

THE pedagogical method used in this book is distinctly different from any found in the usual elementary calculus text. The author, a mathematical master at a boys' preparatory school in England, aims to teach the calculus to the youths by means of the question and answer method. Simple and definite questions on concrete problems concerning matter supposedly familiar to the youthful students are used to develop and fix the fundamental principles of the calculus. There are 180 pages of questions and suggestions; the answers to these cover 100 pages.

An equivalent of a meager high school course in mathematics seems sufficient as a prerequisite. Much of the work could be done orally; a private student might make considerable headway by using the text. Graphical work is minimized and included almost entirely among the answers.

No attempt is made to introduce rigor in the derivation of formulas. For example, the formulas based on the exponential function are developed from a practical consideration of the rate of increase of a sum of money placed at compound interest (continuous)—a concept with which all the students are supposed to be familiar. Or they are advised to draw a figure and use this to derive a formula. Or tables of trigonometric functions may be used to get average rates of increase and thus lead to general formulas. All of which, thoroughly rough and ready, seems like substituting a butcher's cleaver with a fairly dull edge for the scalpel in a surgical operation.

In the integral calculus much time and labor is saved by the following definition of integral: "We have seen that the symbol $D^{-1}f(x)$ denotes the expression for the *amount* of a quantity when its rate of increase is denoted by $f(x)$. The *amount* $D^{-1}f(x)$ is called the **integral** of the function $f(x)$." After which formulas may be applied in large chunks. And there is an everlasting amount of formal differentiating and integrating to be done.

The evaluation of the definite integral is arrived at through

the summation process common to most calculus texts. Much smoother sailing is evident when areas, volumes, centroids and moments of inertia are found.

Needless to say, the law of the mean, extended law of the mean, various forms of remainders and their like are decidedly not included in the chapter on the expansion in series, nor are the various degrees of convergence considered. As the author puts it ". . . there are 'pinnacles' and 'caverns' which only the experienced mathematician should explore."

The concluding chapter is headed The Borderland of Discovery. In other texts this is labeled Approximate Integration.

There is throughout the book much material of a rough and ready nature, which is well worth while and of great service in illustrating the fundamental principles of the calculus; there is certainly lacking the close reasoning which only the use of the method of limits can assure the calculus. The question and answer method might work well with a very limited number of students, though they would certainly have to be English because all units used are intensely British; but, with all the answers given in detail, we doubt very much if even the most conscientious students might not too often be tempted to "look in the book and see."

Ernest W. Ponzer.

Mathematische Abhandlungen, Hermann Amandus Schwarz zu seinem fünfzigjährigen Doktorjubiläum am 6. *August* 1914 *gewidmet von Freunden und Schülern.* Berlin, Springer, 1914. Portrait, viii+451 pp.

This volume forms an imposing testimonial to the influence on the development of modern mathematics of the research and teaching of Schwarz. Of the thirty-four papers contributed by his friends and former students, a majority deal with subjects and methods brought out by him. It is impossible, within the limited space of this review, to do full justice to the rich contents of this volume, so that the reviewer must confine himself to mentioning a few of the papers which have been of particular interest to him, while regretting the necessity of passing in silence many noteworthy contributions.

C. Carathéodory gives a simplification of his former proof in the *Mathematische Annalen*, volume 72, of the most general existence theorem in conformal representation, and establishes

a number of cases in which the continuous one-to-one correspondence of the boundary points may be shown by Schwarz's principle of reflexion. L. Fejér proves some simple and elegant theorems on the convergence of power series, defining a conformal representation, on their convergence circle, and O. Hölder deals with the question of the variation of the solution of a differential equation when the form of the latter is varied. A. Hurwitz solves the problem, proposed by Weierstrass, of the possibility of defining the elliptic sigma function by its addition theorem, while P. Koebe has an article, also inspired by Weierstrass, on analytic functions possessing an algebraic addition theorem. E. Landau carries his researches on prime numbers into definite quadratic forms and pure cubic number fields, Ch. Müntz gives an elegant extension of Weierstrass's theorem on approximation by ordinary polynomials to such as involve non-integral powers, and E. Schmidt presents simple proofs of the fundamental properties of the Newtonian potential, beautiful by their unity of method. J. Schur investigates the expansion of a function in a series of characteristic functions of a positive definite kernel, M. Simon contributes an attractive sketch of the life and works of Sophie Germain, O. Toeplitz gives an example throwing much light upon the scope of Mercer's theorem in integral equations, and in the final paper of the volume, D. Hilbert deals with a general question in the theory of invariants, closely connected with his work in the early nineties.

T. H. Gronwall.

Graphische Methoden. Von C. Runge. Leipzig, Teubner, 1915. iv+142 pp.

This book, which appears as No. 18 of Jahnke's collection of mathematical and physical texts, is a translation of the lectures delivered by the author in 1909–10 at Columbia University and published in 1912 as No. 4 of the publications of the Ernest Kempton Adams Fund for Physical Research.

Chapter I gives the means for performing graphically the four elementary operations on real numbers, the graphical calculation of polynomials in one variable and of linear functions of n variables, including the solution of a system of linear equations, and ends with the representation of complex numbers in the Gaussian plane.

Chapter II deals with graphs of functions of one variable,

the principle of the slide rule, change of variables, the calculation of $z = f(x, y)$ by contour lines of the corresponding surface, and its dual method in line coordinates, the nomography of d'Ocagne. The extension of the latter to more than three variables is briefly indicated. Chapter III contains various methods of graphical integration and differentiation, including the determination of the integral curves of differential equations of the first and second order.

The presentation is concise and very clear, and supported by well chosen illustrative examples and 94 figures, the neatness of which forms a much-needed object lesson to many writers of texts on geometry and graphics.

Regarding literature, there is only a general reference to the corresponding articles in the Encyklopädie; it would have been appropriate to give at least some references for further study, as for instance to d'Ocagne's Calcul graphique et Nomographie, and various papers by Runge, Kutta and others on the graphical integration of differential equations. The book under review brings forth one sad reflection: when will our writers of calculus texts for engineering students see fit to give something really modern and practical on graphical integration and solution of differential equations?

T. H. GRONWALL.

Über die Theorie des Kreisels. Von F. KLEIN und A. SOMMERFELD. Heft I: *Die kinematischen und kinetischen Grundlagen der Theorie.* Zweiter durchgesehener Abdruck. Leipzig, Teubner, 1914. viii+196 pp.

THE second edition of the first part of this standard work differs but slightly from the first one. Literature references have been brought up to date, and occasionally the wording of a theorem is changed.

T. H. GRONWALL.

Konstruktionen in Begrenzter Ebene. Mathematische Bibliothek, herausgegeben von W. LIETZMANN und A. WITTING, XI. Von P. ZÜHLKE. Leipzig und Berlin, B. G. Teubner, 1913. 39 pp. 65 fig.

THIS book treats the subject of constructions in a limited plane primarily from the standpoint of drawing. No restriction is made to a particular set of axioms for proofs, or to any particular set of instruments for constructions. Both metric

and non-metric methods are used, the theorems of Desargues and of Pascal (Pappus) being used for a basis for many of the constructions. Cases in which lines are parallel are considered as distinct from cases in which the lines merely fail to meet on the paper, a distinction which is of special importance in case an instrument for drawing parallels is available.

After a brief historical introduction, the book consists of five subdivisions. I. Unreachable points of intersection of two or more lines. II. Bisection of an angle with non-intersecting sides. III. Construction of triangles and polygons with unreachable vertices. IV. Problems in the theory of circles. V. Bibliography.

A large variety of methods is given, although some of those given as distinct differ in only slight particulars. The chapter on circles, in which points are given by means of non-intersecting circles, i. e., by circles of which parts lie on the paper, but whose points of intersection do not, is especially interesting.

It is much to be regretted that material on such subjects as this is not more readily available in the English language. Perhaps our poverty in well-written elementary books of such a character as to supplement our preparatory-school work is responsible for part of the difficulty in stimulating bright pupils to do work outside of the daily minimum requirement of the textbook. Much of the matter in this little book might well be used for this purpose.

F. W. Owens.

Introduction géométrique à quelques Théories physiques. Par Émile Borel. Paris, Gauthier-Villars, 1914. viii+139 pp.

This book is centered about the theories of relativity and statistical mechanics, and is divided into two distinct parts, of which the first deals in textbook fashion with certain kinematical questions from a purely mathematical point of view, while the second is composed of seven papers, all published before and rather loosely connected with each other, dealing with various topics in mathematical physics in a critical and philosophical manner. The first part contains four chapters on the euclidean displacements in two and three dimensions, the four-dimensional euclidean geometry, a two-dimensional hyperbolic geometry, and the three- and four-dimensional hyperbolic displacements and their application to the kine-

matics of relativity. A fifth chapter, on functions of a very large number of variables, and areas and volumes in a geometry of 10^{24} dimensions, leads up to statistical mechanics, the number stated being of the order of magnitude of the number of molecules in the unit volume, or the number of dimensions of their velocity space.

The titles of the seven papers forming the second part are as follows: On the principles of the kinetic gas theory; statistical mechanics and irreversibility; the relativity of space according to Henri Poincaré; some remarks on the theory of resonators; on a problem in geometric probability; the kinematics of the theory of relativity; molecular theories and mathematics.

These investigations of some of the most modern questions in theoretical physics should prove of great interest to both mathematicians and physicists.

T. H. GRONWALL.

Grundzüge der Geodäsie. Von M. NÄBAUER. Leipzig, Teubner, 1915. xiv+420 pp.

THIS book forms volume 3 of Handbuch der angewandten Mathematik, edited by H. E. Timerding, and is written primarily with the purpose of acquainting students of mathematics with the modern methods of geodesy. This purpose is quite successfully accomplished by presenting just enough of the practical side of the subject to give the proper setting for the clear and terse mathematical discussion of the underlying principles and the sources of error in the various geodetic operations.

The first part contains the theory of errors and the application of the method of least squares to the reduction of observations. Part two, plane surveying, deals with the surveying instruments, the various kinds of field work (the paragraph on photogrammetry is especially well done), plotting and computation of areas. Part three, higher geodesy, begins with triangulation and the various kinds of coordinates on the earth considered as a sphere, proceeds to the earth ellipsoid, its conformal representation on the sphere and the determination of its dimensions, and ends with a brief account of the determination of the exact figure of the earth by astronomical and pendulum observations.

The mathematical apparatus is confined to the elements of the calculus, and the volume contains much that could be used

to advantage in bringing a course in trigonometry in closer touch with one of its main applications.

T. H. GRONWALL.

Relativity. By A. W. CONWAY. Edinburgh Mathematical Tracts, number 3. London, G. Bell and Sons, 1915. 2 s. 43 pp.

THIS tract is a course of four lectures delivered before the Edinburgh Mathematical Colloquium on the subject of relativity. The audience were representative of various branches of science. These four lectures start with fundamentals, followed by a study of the transformation of the electromagnetic equations, applications to radiation and electron theory, and Minkowski's transformation. The lecturer has succeeded very well in presenting the essentials of the relativity hypothesis free from metaphysics, and speculations of any kind. He has a decidedly sane treatment. There are examples enough to make the ideas clear, stated in everyday terms, and not in terms of the usual mathematical model. It is a serviceable introduction.

JAMES BYRNIE SHAW.

A Theory of Time and Space. By ALFRED A. ROBB. Cambridge, University Press, 1914. vi + 373 pp.

THIS treatise is an elaboration of a previous publication on the same subject. In brief it is an analysis of space and time relations by means of a single type of order called conical. The author also calls the result optical geometry. The treatment is of an axiomatic character, the few diagrams serving only as schemes. There are twenty-one postulates set down, and from these and various definitions, some two hundred and six theorems are deduced. These ultimately lead to statements which permit an algebraic formulation by the use of four parameters, which may be interpreted as the usual x, y, z, and t, the last having a somewhat different rôle from the others. The notion of relativity of course hovers in the background, but any one seeking light on that notion here will be disappointed, as the book is simply a development of a very abstract geometry of four dimensions.

It is not possible to give a resumé of the contents in a review, but some idea can be gained of the point of view by stating

the fundamental conceptions of the author in a somewhat different and more picturesque manner than he does. The model thus given is of course due to the reviewer and not to the author. Let us surround every point of ordinary three-dimensional space with a sphere, the radii being anything we like, and some radii being considered to be positive, some negative. Then any such sphere with its center may be called an element. We will consider that if all the radii are increased by the same amount the elements are unchanged. Two different spheres about the same center at one time are two distinct elements. It is in this conception that the term conical order becomes appropriate, for if we start with a two-dimensional space and surround the points by circles, when the radii are increased by the same amount we may move the plane of operations parallel to itself a proportional amount, thus generating cones by the expanding circles. This conception however cannot be used in the imagery of three dimensions.

The element B is said to be *after* the element A when the distance of the center of A from that of B is less than the algebraic difference of the radii of B and A, this difference being positive. If the distance between the centers is less than the difference between the radii of A and B, this difference being positive, B is said to be *before* A. If the distance between centers is greater than the absolute value of the difference of the radii neither A nor B is before or after the other. This is the type of order upon which the author bases his whole development. The appropriateness of the terms from an optical point of view is seen if we consider the spheres to be the wave-fronts of light signals from the points. For a signal to proceed from A to B within a given time, the element B must be after the element A.

The α-subset of A consists of all elements whose centers and radii are such that the distances from the center of A to the other centers are equal to the differences between the radius of A and the radii of the other elements. The β-subset of A consists of all elements whose distances from the center of A equal the differences of their radii and the radius of A. The element A belongs to both subsets. If the radii are all increased sufficiently the α-subset will consist of all elements whose spheres are internally tangent to that of A, or would become so by further increase of radii.

The elements A_1 and A_2 determine an *optical line* if A_2 is an element of either the α-subset of A_1 or its β-subset. The optical line then consists of all elements which are in either the two α-subsets of A_1 and A_2 or their two β-subsets. All these elements will be tangent to the sphere of A_1 at the same point, or will become so when the radii are all sufficiently increased. If A_2 is neither before nor after A_1, then the two determine a *separation line,* because the elements which have centers on this line may have spheres of such size that no one is before or after any other. In this sense they are like particles on a line in space. If A_2 is not in the α-subset of A_1 but is after A_1 they determine an *inertia line,* in the sense that a particle properly chosen could move along this line with a given velocity, that is to say elements may be so chosen as to be successively after A_1 and before A_2. A separation segment is said to have a length r, which is the second side of a right triangle made on the distance between centers as hypotenuse, and the difference of the radii as first side. An inertia line is said to have a length which is the second side of a right triangle whose hypotenuse is the difference of radii and first side the distance between centers. In the latter case however the unit of measure is taken to be a standard number v called the velocity of light, and which would be the rate of increase of all the radii of all the elements.

It is clear that the time deduced in such an axiomatic treatment is not the Time of philosophy nor of psychology, but is merely a kind of order. The time involved here does not flow, for the particular system of spheres we chose is a stationary set, and any increase in radii is for convenience merely, since only the differences of the radii are ever considered. Time enters only by constructing a second system of spheres about all the points of space. The new radii are then representative of times (instants) different from those represented by the first set. As the author claims, he is studying a type of order, which permits of the abandonment of the notion of simultaneity save as a local phenomenon. In this sense he is studying relativity. The treatise will be interesting in the main to students of postulational geometries.

JAMES BYRNIE SHAW.

Repertorium der Physik. Von R. H. Weber und R. Gans. Erster Band: *Mechanik und Warme.* Erster Teil. Leipzig, B. G. Teubner, 1915. xii + 434 pp. 8vo. Price, 8 Marks.

This is the first part of the first volume of a repertorium of physics similar to the well-known Pascal repertorium of mathematics. The general character is more that of a small treatise, intermediate between an ordinary textbook and a manual, since it is written in a fairly connected style. It includes many results of memoirs, however, that one would not find in the ordinary text, and omits many details, particularly numerical ones, that would be found in a complete manual. In this way it has been possible to compress within a reasonable space a quite convenient vademecum for both physicists and mathematicians.

This first part is divided into three books. The first and second are written by R. Gans of La Plata. The third book is written by F. A. Schulze. The first book is on the mechanics of discrete particles. A list of the chapter headings will convey a sufficient idea of the contents. These are: Fundamentals of mechanics; Principles of mechanics; Dynamics of rigid bodies; Gravitation; Coordinate systems, rotation of the earth, centrifugal force; Friction; Vibrations. The second book is devoted to the mechanics of continuous media under two divisions: *A*, Elasticity; *B*, Hydrodynamics. Under *A* we find chapters on: Kinematics and dynamics of deformable media; Statical problems of the theory of elasticity; Dynamical problems of the theory of elasticity. Under *B* we find the chapters treating of: Equations of motion and general theorems; Kinematics of fluids with an immersed fixed body; Dynamics of fluids containing an immersed fixed body; Problems in two dimensions; Waves; Vibrations of the air; Viscosity; Tides. The book on Acoustics contains chapters on: Propagation of sound; Intensity of sound; Various problems; Musical scales.

The second part of the first volume will treat of Capillarity, Heat, Statistical mechanics, Kinetic theory of gases; the second volume will be devoted to Electricity, Magnetism, and Optics.

It may be regretted that for a subject as extensive as physics, the work is not a little more comprehensive, but as it is it will be useful.

James Byrnie Shaw.

NOTES.

The April number of the *Transactions of the American Mathematical Society* contains the following papers: "On multiform solutions of linear differential equations having elliptic function coefficients," by W. L. Miser; "On the foundations of plane analysis situs," by R. L. Moore; "On the generalized Jacobi-Kummer cyclotomic function," by H. H. Mitchell; "Proof of a theorem of Haskins," by D. Jackson; "On the measurable bounds and the distribution of functional values of summable functions," by C. N. Haskins; "Jacobi's condition for problems of the calculus of variations in parametric form," by G. A. Bliss.

The twenty-first summer meeting and eighth colloquium of the American Mathematical Society will be held at Harvard University during the week beginning Monday, September 4, 1916. The first two days will be devoted to the regular sessions for the presentation of papers. The colloquium will open on Wednesday morning and close on Saturday morning. Two courses of five lectures each will be given as follows (the list of principal topics is appended):

Professor G. C. Evans: "Topics from the theory and applications of functionals, including integral equations."

The lectures will attempt a brief survey of the present state of this theory, showing how its results have been applied in various branches of analysis and in physics. The following topics will be considered, with the purpose of keeping the point of view as general as possible:

Functions depending on curves and surfaces in three dimensions, determination of Green's function by means of variational equations; functions depending on curves and surfaces in four dimensions, integrals of analytic functions of two complex variables; the linear functional relation, the integral equation of the third kind; applications of the Volterra theory of functional relations; applications of the Hilbert theory of integral equations.

Professor Oswald Veblen: "Analysis situs."

This course will attempt to give an account of the present state of this elementary but relatively undeveloped branch of geometry. Among the topics considered will be:

The n-dimensional cell; separation of a cell into regions by polyhedra; combinatorial properties of polyhedra; manifolds as generalized polyhedra; abstract equivalence of manifolds; numerical invariants and group of a manifold; equivalence of two manifolds within a third, theory of knots; continuous transformations of a manifold into itself.

The chief references can be obtained from the article on Analysis Situs in the Encyklopädie and the chapter on Topology in the new edition of Pascal's Repertorium.

At the meeting of the London mathematical society held March 9 the following papers were read: By P. A. Macmahon: "Some applications of general theorems of combinatory analysis"; by H. F. Baker: "Mr. Grace's theorem on six lines with a common transversal"; by H. E. J. Curzon: "The integrals of a certain Riccati equation connected with Halphen's transformation"; by Hilda P. Hudson: "A certain plane sextic"; by W. P. Milne: "The construction of co-apolar triads on a cubic curve"; by J. Bondman: "The dynamical equations of the tides."

At the meeting of the Edinburgh mathematical society on March 10 the following papers were read: By J. F. Tinto: "Transformations founded on the space cubic and its chord system"; by J. Dougall: "Elliptic cylindrical harmonics."

The Association of mathematics teachers of New Jersey held its fourth regular meeting at Princeton University on May 6, 1916. The programme included the presidential address of Professor H. B. Fine: "The theory of incommensurable magnitudes as set forth in the tenth book of Euclid's Elements;" Report of the committee on trigonometry courses; Edwin Florance: "Ptolemy's theorem;" E. S. Ingham: "An exposition of Napier's principle of logarithms;" Rev. F. C. Doan: "Certain religious implications of the mathematical infinite;" J. C. Stone: "The ultimate aim of a course in arithmetic."

The Paris academy of sciences announces as the subject for its Grand prize in mathematics (3000 francs) for 1917 the following:

"To perfect in an important point the theory of successive

powers of a given substitution, the exponent of the power increasing indefinitely. The influence of the initial element is to be considered, and the consideration may be limited to the simplest cases, such as that of rational substitutions in one variable."

The Academy has elected as corresponding members Professors LIAPOUNOF, of Petrograd, and C. J. DE LA VALLEÉ POUSSIN, of Louvain.

ON account of the war two mathematical periodicals have suspended publication, viz., *L'Education Mathématique*, which concluded its sixteenth and last volume with the issue for July, 1914, and the *Revue de Mathématiques Spéciales*, last issued in September, 1914, in completion of its twenty-fourth consecutive year.

THE following university courses in mathematics are announced for the academic year 1916–1917:

CORNELL UNIVERSITY.—By Professor J. McMAHON: Theory of probabilities, three hours.—By Professor V. SNYDER: Descriptive geometry (first term), three hours; Algebraic curves and surfaces, three hours.—By Professor F. R. SHARPE: Theory of potential and Fourier series, three hours.—By Professor W. B. CARVER: Differential geometry (first term), three hours; Theory of numbers (second term), three hours. —By Professor A. RANUM: Modern algebra (second term), three hours.—By Professor D. C. GILLESPIE: Principles of mechanics, three hours.—By Professor W. A. HURWITZ: Theory of functions of real variables, three hours.—By Professor C. F. CRAIG: Functions of a complex variable, three hours.—By Professor F. W. OWENS: Differential equations, three hours; Mathematical physics, three hours.—By Dr. J. V. McKELVEY: Advanced calculus, three hours.—By Dr. L. L. SILVERMAN: Analytic geometry, three hours.—By Dr. M. G. GABA: Projective geometry, three hours.—By Mr. H. BETZ: Graphical processes and numerical calculation, three hours.

HARVARD UNIVERSITY.—All courses meet three times a week throughout the year, except those marked*, which meet for half a year.—By Professor W. F. OSGOOD: Introduction to potential functions and Laplace's equation;* Galois's theory

of equations.*—By Professor M. Bôcher: Interpolation and approximation;* Theory of functions; Linear differential equations, complex variables.*—By Professor C. L. Bouton: Advanced calculus; Geometrical transformations, with special reference to the work of Sophus Lie.—By Professor J. L. Coolidge: Modern geometry and modern algebra; Line geometry.—By Professor E. V. Huntington: Fundamental concepts of mathematics.*—By Professor G. D. Birkhoff: Dynamics, second course;* Analytical theory of heat, Fourier's series, and Legendre's polynomials;* Applications of the calculus of variations.*—By Professor D. Jackson: Infinite series and products;* Lebesgue integrals.*—By Dr. G. M. Green: Elementary differential equations;* Differential geometry of curves and surfaces.—By Dr. E. Kircher: Vector analysis;* Finite groups.*—By Mr. W. LeR. Hart: Calculus of variations;* Functions of infinitely many variables.*

Professors Osgood and Birkhoff will conduct a fortnightly seminar in analysis.

Courses of research are also offered by Professor Osgood in the theory of functions, by Professor Bôcher in the real solutions of linear differential equations, by Professor Bouton in the theory of point-transformations, by Professor Coolidge in geometry, by Professor Birkhoff in the theory of differential equations, by Professor Jackson in the theory of functions of a real variable, and by Dr. Green in differential geometry.

University of Pennsylvania.—By Professor E. S. Crawley: Modern analytic geometry.—By Professor G. E. Fisher: Theory of functions of a complex variable.—By Professor G. H. Hallett: Galois theory of equations.—By Professor F. H. Safford: Partial differential equations.—By Professor M. J. Babb: Introduction to modern higher algebra.—By Professor O. E. Glenn: Theory of invariants.—By Professor H. H. Mitchell: Elliptic functions.—By Dr. R. L. Moore: Functions of a real variable with an introduction to certain phases of general analysis.—By Dr. F. W. Beal: Differential geometry.

Princeton University (1916–1917).—By Professor H. B. Fine: History of analysis, second term, three hours.—By Professor L. P. Eisenhart: Differential geometry, three hours.—By Professor Oswald Veblen: Projective geometry,

three hours; Seminar.—By Professor PIERRE BOUTROUX: Analysis, three hours; Differential equations, three hours.—By Dr. A. A. BENNETT: Algebra, three hours; Projective geometry, three hours.—By Dr. J. W. ALEXANDER: Newtonian potential function, three hours.

SHELDON travelling fellowships for the year 1916–1917 have been awarded by Harvard University to Mr. R. W. BRINK and Mr. J. L. WALSH. During the current academic year a similar fellowship is held by Mr. L. R. FORD, who is studying in Paris. Mr. A. L. MILLER holds a Rogers travelling fellowship and is studying in Turin.

ONE of the two fellowships, of the annual value of seven hundred and fifty dollars, in the department of mathematics of the Rice Institute remains to be filled for the academic year 1916–1917. The successful candidate will be expected to enter upon a course of study and research work leading to the degree of doctor of philosophy, and also to assist with elementary teaching of mathematics for six hours per week. The fellows will be able to live in the residential hall of the Institute, where board and lodging will be provided for them at about thirty dollars per month. Applications accompanied by testimonials and a full statement of previous work and training should be addressed to the Department of Mathematics, Rice Institute, Houston, Texas.

ON March 16, 1916, the Scandinavian mathematicians celebrated at Stockholm the seventieth birthday of their illustrious colleague, Professor G. MITTAG-LEFFLER, founder and director of the *Acta Mathematica*. Many messages of congratulation were received from mathematicians of other countries. On this occasion Professor Mittag-Leffler and his wife set aside their entire fortune for the foundation of an International institute for pure mathematics.

MR. R. E. GILLMAN, of Princeton University, has been appointed instructor in mathematics at Cornell University.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BACKES (W.). Ein Nachtrag zu den Beweisen für den Fermatschen Satz $x^n + y^n = z^n$. Mainz, 1915. 4 pp. M. 3.00

BELL (E. T.). Arithmetical theory of certain numerical functions. (University of Washington Publications in Mathematical and Physical Sciences, vol. 1, No. 1.) Seattle, University of Washington, 1915. 4to. 44 pp. $0.50

BÔCHER (M.). Syllabus of a brief course in solid analytic geometry. (Rectangular coordinates.) Cambridge, Mass., Harvard Cooperative Society, 1916. 8vo. 10 pp. Paper. $0.15

BONYNGE (W.). See FISHER (A.).

FISHER (A.). The mathematical theory of probabilities and its application to frequency curves and statistical methods. Translated and edited from the author's original Danish notes with the assistance of W. Bonynge, with an introductory note by F. W. Frankland. Vol. 1. Mathematical probabilities and homograde statistics. New York, Macmillan, 1915. 20+171 pp. $2.00

FORD (W. B.). Studies on divergent series and summability. (University of Michigan Studies. Scientific Series, vol. 2.) New York, Macmillan, 1916. 4to. 12+194 pp. Cloth. $2.50

FRANKLAND (F. W.). See FISHER (A.).

FUBINI (G.). Lezioni di analisi matematica. 2a edizione, interamente rifusa. (Grande biblioteca tecnica, no. 9.) Torino, Soc. tip. ed. Nazionale, 1915. 8vo. L. 14.00

LEWIS (F. P.). Geometrical application of the theory of the binary quintic. (Diss.) Baltimore, Johns Hopkins, 1914. 4to. 24 pp.

RULF (W.). Analytische Geometrie für höhere Gewerbeschulen. Wien, Deuticke, 1915. 158 pp. Geb. Kr. 3.20

SPOLTORE (N.). La duplicazione grafica del cubo. Vasto, tip. Zaccagnini, 1915. 8vo. 8 pp. con tavola.

UDZIELA (E.). Neue Lösung des Fermatschen Problems $x^n + y^n = z^n$. Wien, Udziela, 1915. 6 pp.

II. ELEMENTARY MATHEMATICS.

ANDERSON (R. F.). See PHILIPS (G. M.).

BEHRENDSEN (O.) und GÖTTING (E.). Lehrbuch der Mathematik nach modernen Grundsätzen. Leipzig, Teubner, 1915. 8vo. Ausgabe B für Schulen mit realistischem Lehrplan. Unterstufe. 3te Auflage. 8+350 pp. Oberstufe. 2te Auflage. 7+404 pp. M. 2.80+4.00

BENEDICT (H. Y.) and CALHOUN (J. W.). Teaching of plane geometry. 2d issue. (Official series, No. 104.) Austin, University of Texas, 1914. 8vo. 57 pp.

Betz (W.) and Webb (H. E.). Solid geometry. With the editorial co-operation of P. F. Smith. Boston, Ginn, 1916. Pp. 22+327–504. $0.75

Brookman (T. A.). A practical algebra for beginners. New York, Scribner, 1915. 8vo. 17+322 pp. $1.12

Calhoun (J. W.). See Benedict (H. Y.).

Courtis (S. A.). Courtis standard practice tests in the four operations with whole numbers. (Efficiency drill series.) Three parts. Yonkers-on-Hudson, World Book Company, 1914. Teacher's manual, paper, $0.50. Practice tests (48 lesson leaves), $0.16. Student's record, per package of 50, $1.00.

Fischer (P. B.). Rechenbuch für die unteren und mittleren Klassen der höheren Knabenschulen. Leipzig, Teubner, 1915. Ausgabe in 1 Bande. 4+196 pp. M. 2.20. Ausgabe in 3 Heften. 1ter Teil: Lehrstoff der Sexta. 4+73 pp. M. 1.00. 2ter Teil: Lehrstoff der Quinta. 4+62 pp. M. 0.80. 3ter Teil: Lehrstoff der Quarta. 4+60 pp. M. 0.80.

Gifford (J. B.). Everyday arithmetic. Boston, Little, Brown and Company, 1916. $0.35

Götting (E.). See Behrendsen (O.).

Hamilton (S.). Key to Hamilton's Arithmetics, second and third books; New Jersey edition. New York, American Book Company, 1915. 188 pp. $0.64

Hart (W. W.). See Wells (W.).

Hartl (H.). Lehrbuch der Planimetrie. 3te Auflage. Wien, Deuticke, 1915. 143 pp. Geb. Kr. 3.00

Harvey (L. D.). Essentials of arithmetic; with everyday problems relating to agriculture, commerce, and other vocations. (Three book series.) 2d book, parts 1–2. New York, American Book Company, 1915. $0.36+0.36

Holloway (H. V.). Experimental study to determine the relative difficulty of the elementary number combinations in addition and multiplication. Bordentown, N. J., Holloway, 1915. 8vo. 102 pp. $1.00

Hoyt (F. S.) and Peet (H. E.). Everyday arithmetic. 3 books. Boston, Houghton-Mifflin, 1915. 138 +140+190 pp. $0.40+0.40+0.45

Jacob (J.). Arithmetik. 3ter Teil: Oberstufe. 2te Auflage. Wien, Deuticke, 1915. 111 pp. Schlüssel dazu. 84 pp. Geb. Kr. 2.20+2.40

Jessup (W. A.). Economy of time in arithmetic. (University of Iowa extension bulletin, No. 5.) 8vo. Pp. 461–476. Paper. Gratis.

Millis (J. F.). See Stone (J. C.).

Milne (W. J.). Key to New York state arithmetic; first and second books. New York, American Book Company, 1915. 304 pp. $0.80

Miner (G. W.). See Moore (J. H.).

Mlodzievsky (B.). Rapport sur l'enseignement mathématique aux cours supérieurs des femmes à Moscou. (Commission internationale de l'enseignement mathématique. Sous-commission russe.) Petrograd, Luinik, 1915. 20 pp.

MOORE (J. H.) and MINER (G. W.). Concise business arithmetic. Boston, Ginn, 1915. 8vo. 6+283 pp. $0.75

MUNDT (J.). Rechenaufgaben über den Weltkrieg für die deutsche Jugend. Cöln, P. Schmitz, 1915. 24 pp. M. 0.10

OBERG (E. V.). Elementary algebra. (Machinery's reference series, No. 138.) New York, Industrial Press, 1914. 40 pp. Paper. $0.25

PEET (H. E.). See HOYT (F. S.).

PENDLETON (F. T.). Modern arithmetic. New London, Conn., F. T. Pendleton, 1914. 37 pp. $0.25

PHILIPS (G. M.) and ANDERSON (R. F.). Course of study in arithmetic for grades below the high school. Boston, Silver, Burdett and Company, 1915. 45 pp. Gratis.

RUBINSTEIN (A.). Review in algebra for those who know algebra but want to know it well enough to pass an examination. Revised edition. New York, Hinds, Noble and Eldridge, 1915. 60 pp. $0.25

SMITH (P. F.). See BETZ (W.).

SOMERVILLE (F. H.). Key to elementary algebra. Revised edition. New York, American Book Company, 1915. 374 pp. $1.00

STONE (J. C.) and MILLIS (J. F.). Manual and course of study for teachers of arithmetic. Chicago, Sanborn, 1914. 38 pp. $0.10

WAGNER (E.). Repetitorium der Mathematik. 2ter Teil: Trigonometrie, Stereometrie, Maxima und Minima. Strassburg in Els., Van Hauten, 1914. 92 pp.

WEBB (H. E.). See BETZ (W.).

WEINTROB (R.). Silk arithmetic. New York, Simmons, 1915. 47 pp. $0.35

WELLS (W.) and HART (W. W.). Plane and solid geometry. Boston, Heath, 1916. 8vo. 10+467 pp.

III. APPLIED MATHEMATICS.

ANNUAIRE astronomique et météorologique. Paris, Flammarion, 1916. 431 pp.

ANNUAIRE pour l'an 1916 publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1915. 6+502 pp. Fr. 1.50

ANNUARIO del observatorio de Madrid para 1916. Madrid, Bailly-Balliere, 1916. 645 pp.

ASHWORTH (J. R.). An introductory course of practical magnetism and electricity. 3d edition. London, Whittaker, 1916. 17+96 pp. 2s.

AUERBACH (F.). Die Physik im Kriege. 2te Auflgae. Jena, Fischer, 1915. 4+209 pp. M. 4.00

BOWLEY (A. L.). Elementary manual of statistics. (Modern commercial text-books.) New York, Scribner, 1914. 8vo. 215 pp. $2.00

DADOURIAN (H. M.). Analytical mechanics. 2d edition. New York, Van Nostrand, 1915. $3.00

ERWIN (M.). The universe and the atom. London, Constable, 1916. 8vo. 8s. 6d.

EVANS (C. T.). Arithmetic of elementary chemistry. Pottstown, Pa., Hill School, 1914. 99 pp. $1.00

FREUCHEN (P. B.). Termodynamik. Kjöbenhavn, Lehmann, Stages Forlag, 1916. 144 pp.

HALER (P. J.) and STUART (A. H.). A first course in engineering science. London, University Tutorial Press, 1915. 8+191 pp. 2s. 6d.

HAWKINS (N.). Self-help mechanical drawing; an educational treatise. New York, Audel, 1915. 8vo. 13+299 pp. $2.00

HUTTON (P. W.). Mechanical drawing for industrial and continuation schools. Chicago, Scott, Foresman and Company, 1915. 8vo. 176 pp. $0.90

KOITCAMP (J. P.). Elementary mechanics for the practical engineer; engineer's study course from power. New York, McGraw-Hill, 1915. 8vo. 7+181 pp. $1.50

LEEDS (C. C.). Mechanical drawing for technical and high schools. 3d edition. New York, Van Nostrand, 1915. $1.25

MAILLOUX (C. O.). See SEMENZA (G.).

MASO (R. P. M. S.). Historia del observatorio de Manila, 1865–1915. Manila, E. C. McCullaugh, 1915. 210 pp.

MEYER (A. F.). Graphs of meteorological and hydrological data. Minneapolis, Northwestern School Supply, 1915. 8vo. 47 pp. Paper. $0.15

MILLER (D. C.). The science of musical sounds. New York, Macmillan, 1916. $2.50

PÜNING (H.). Lehrbuch der Physik. 10te Auflage. Münster, Aschendorff, 1915. 359 pp. M. 3.70

SCHRAIDT (F. F.). Geometrical drawing; a selection of plates for practical use in elementary mechanical drawing. San Francisco, Whittaker and Ray-Wiggin, 1915. 8vo. 78 pp. $0.65

SEMENZA (G. and M.). Graphical determination of sags and stresses for overhead line construction. Translated from the Italian by C. O. Mailloux. New York, McGraw-Hill, 1915. 10+24 pp. $3.00

SEMENZA (M.). See SEMENZA (G.).

SMITH (E.). Evolution of a gravitating, rotating, condensing fluid. (University of Cincinnati Studies.) Cincinnati, 1915. 45 pp. Paper. $0.50

STUART (A. H.). See HALER (P. J.).

THOMAS (C. M.). Compass surveying and the simplified calculation of farm areas. Wytheville, Pa., C. M. Thomas, 1915. 8vo. 6+92 pp. $2.00

WEICH (C. W.). Elementary mechanical drawing; theory and practice, with chapters on geometrical drawing, mensuration, and reproduction of drawings: a text-book for technical, secondary, trade and vocational schools. New York, McGraw-Hill, 1915. 8vo. 10+250 pp. $1.75

THE APRIL MEETING OF THE SOCIETY AT CHICAGO.

The thirty-seventh regular meeting of the Chicago Section, being the sixth regular meeting of the Society at Chicago, was held at the University of Chicago, on Friday and Saturday, April 21 and 22. The meeting was attended by about sixty-five persons, among whom were the following forty-six members of the Society:

Professor R. M. Barton, Dr. Josephine E. Burns, Professor D. F. Campbell, Professor R. D. Carmichael, Dr. E. W. Chittenden, Dr. G. R. Clements, Professor H. E. Cobb, Professor D. R. Curtiss, Professor S. C. Davisson, Dr. W. W. Denton, Professor L. E. Dickson, Professor A. Dresden, Professor W. B. Ford, Professor Tomlinson Fort, Professor A. B. Frizell, Professor A. S. Hathaway, Professor E. R. Hedrick, Dr. Cora B. Hennel, Professor W. C. Krathwohl, Dr. W. V. Lovitt, Professor A. C. Lunn, Professor W. D. MacMillan, Dr. T. E. Mason, Professor G. A. Miller, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Dr. A. L. Nelson, Professor C. I. Palmer, Dr. A. Pell, Professor H. L. Rietz, Professor W. H. Roever, Professor D. A. Rothrock, Miss I. M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor C. H. Sisam, Professor H. E. Slaught, Professor R. B. Stone, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. VanVleck, Professor E. J. Wilczynski, Professor K. P. Williams, Dr. C. H. Yeaton, Professor J. W. A. Young.

The sessions were presided over by Professor W. B. Ford, chairman of the Section, relieved during the session of Saturday forenoon by Professor E. R. Hedrick, vice-president of the Society. Forty-four persons were present at the dinner on Friday evening.

The following papers were presented at this meeting:

(1) Professor D. M. Smith: "Jacobi's condition for the problem of Lagrange in the calculus of variations."

(2) Professor C. H. Sisam: "On a configuration on certain surfaces."

(3) Professor T. R. Running: "A new method for deriving weir formulas."

(4) Professor Arnold Emch: "A new configuration on an elliptic cubic and its group of order 16."

(5) Professor L. L. Dines: "A characteristic property of self-projective curves."

(6) Dr. A. L. Nelson: "Plane nets with equal invariants."

(7) Dr. A. L. Nelson: "Quasi-periodic asymptotic plane nets."

(8) Dr. E. W. Chittenden: "A theorem in general analysis."

(9) Dr. E. W. Chittenden: "On the equivalence of écart and voisinage."

(10) Professor R. D. Carmichael: "On the asymptotic character of functions defined by series of the form $\Sigma c_n g(x + n)$."

(11) Professor W. D. MacMillan: "A theorem relating to irrational numbers."

(12) Professor W. D. MacMillan: "A reduction of certain differential equations of the second order to algebraic types."

(13) Professor Tomlinson Fort: "A class of developments in orthogonal functions."

(14) Dr. R. L. Borger: "On the Cauchy-Goursat theorem."

(15) Professor H. T. Burgess: "A practical method for determining elementary divisors."

(16) Dr. J. O. Hassler: "Plane nets periodic of period 3 under the Laplacian transformation."

(17) Professor A. S. Hathaway: "The expansion, in terms of the coefficients of an equation, of the homogeneous products of the roots as a whole, and when restricted to k roots in each term."

(18) Professor W. B. Ford: "A theorem in the calculus of residues."

(19) Professor G. A. Miller: "Graphical method of finding the possible sets of independent generators of an abelian group."

(20) Professor A. B. Frizell: "Postulates of continuity for arithmetic."

(21) Professor K. P. Williams: "Concerning Hill's derivation of the Lagrange equations of motion."

(22) Professor Henry Blumberg: "On convex functions."

(23) Professor HENRY BLUMBERG: "On a theorem of Kempner concerning transcendental numbers."

(24) Professor E. B. VAN VLECK: "On the composition of non-loxodromic substitutions."

(25) Mr. TOBIAS DANTZIG: "A geometrical treatment of plane transformations."

(26) Professor E. H. MOORE: "On a definition of the concept: limit of a function."

(27) Mr. A. ELMENDORF: "A differentiating machine."

(28) Mr. A. M. HARDING: "On certain loci projectively connected with a given plane curve."

(29) Miss PAULINE SPERRY: "Properties of a certain projectively defined two-parameter family of curves on a general surface."

(30) Dr. C. H. YEATON: "Surfaces characterized by certain special properties of their directrix congruences."

Dr. Hassler, Mr. Harding and Miss Sperry were introduced by Professor Wilczynski, Mr. Dantzig by Professor Davisson, and Mr. Elmendorf by Professor Dresden. The papers of Professors Smith, Running, Dines, Burgess and Blumberg, and Dr. Borger were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. For the problem of Lagrange, in the calculus of variations, the Euler rule, Weierstrass condition, and corner point condition are obtainable without the use of the second variation. The Jacobi condition, however, is derived by the use of complicated transformations of the second variation or with the exclusion of important special cases. In the paper of Professor D. M. Smith it is shown that the well-known necessary condition for a minimum—i. e., that the second variation must be positive or zero for every set of admissible variations—implies a problem of Lagrange of precisely the same type as the original problem. An application of the Euler rule, Weierstrass condition, and corner point condition to this new problem leads to simple and inclusive proofs of the Legendre and Jacobi conditions for the original problem. The paper will appear in the *Transactions*.

2. Professor Sisam's paper appeared in full in the May BULLETIN.

3. The following method for deriving the formulas for the rate of flow of water over weirs avoids the necessity of a large number of experiments and can be applied to weirs of any head.

A tank having a constant horizontal cross-section has a weir fitted with a gate which can easily be removed to allow the flow of water. A chronograph connected electrically with a float and also with a vibrating pendulum records time to hundredths of a second and head to thousandths of a foot. Professor Running experimented with a right-angled notch weir, head from 2 ft. to 1.7 ft., and obtained for the law connecting time and head

$$T = -23 + 62.2H^{-1.47}.$$

T and H must satisfy the relation

$$QdT = -SdH,$$

where Q represents the rate of outflow and S the horizontal cross-section of the tank. From these two equations the formula for the weir was found to be

$$Q = 2.52H^{2.47}.$$

This represents quite closely the results of experiments performed by other methods.

The above is part of a paper to be published jointly with Professor King.

4. It is well known that a plane cubic may be generated in an infinite number of ways by projective quadratic pencils of straights with a self-corresponding element. The vertices of these pencils are points of a Steinerian couple. A Steinerian quadruple on an elliptic cubic is formed by the points of tangency of the four tangents from a point of the elliptic cubic to the same cubic. The quadruple contains six couples, and, consequently, six projectivities of quadratic pencils may be formed which may be written in 12 different ways, and which all generate the same cubic. Applying a certain cyclic process to the equations of these projectivities and making use of a certain theorem in algebra, Professor Emch obtains a configuration of 48 straights, which in triplets are concurrent

in 16 points. From one triplet all the others are obtained by the substitutions of a group of order 16 with 15 subgroups of order 2. The paper will be published in the *Annals of Mathematics*.

5. Self-projective curves, or W-curves, were defined by Klein and Lie to be those curves which admit infinitesimal projective transformations into themselves. In this paper, Professor Dines determines for these curves a simple characteristic property somewhat similar to the Steinerian definition of conics. The paper will appear in the *Annals of Mathematics*.

6. Wilczynski has treated the projective differential geometry of plane nets, as well as that of curved surfaces. Dr. Nelson sets up, in this paper, a perspective connection between these two theories by means of the following theorem of Koenigs: The asymptotic curves of a surface are projected from a fixed point on a fixed plane in a net with equal Laplace-Darboux invariants, and conversely, a plane net with equal invariants may be considered as the perspectives, from a fixed point, of the asymptotic curves of a surface. The paper will be offered for publication to the *Palermo Rendiconti*.

7. Dr. Nelson's second paper appears in the present number of the BULLETIN.

8. Dr. Chittenden completes the proof of the theorem: Every class $\mathfrak{M}$ with the composite property $D_1K_{12}\Delta$ has the property K_{12*}, and $\mathfrak{M}_*$ has the composite property

$$LCD_1A\Delta K_{12}K_{12*}.$$

The proof is accomplished with the aid of a theorem of Professor A. D. Pitcher which differs from the present theorem in that the hypothesis contains the additional property B_3. B_3 is a special property defined in terms of Δ and boundedness from zero.

Professor Pitcher in a discussion of the complete existential theory of the eight properties L, C, D, A, K_1, K_2, Δ, K_{12*}, left unsettled the question of the existence of three composite properties; viz.,

$$L^-C^-DA\Delta K_1K_2{}^-K_{12*}; \qquad LC^-DA\Delta K_1K_2{}^-K_{12*};$$

$$LC^-D^-A\Delta K_1K_2{}^-K_{12*}.$$

The theorem of this paper shows that the first two are non-existent, and that the third is non-existent for any system $(\mathfrak{P}; \Delta; \mathfrak{M}^{D_1})$. This extends the results obtained by Professor Pitcher who showed that the three properties were non-existent for any system $(\mathfrak{P}; \Delta; \mathfrak{M}^{D_1B_3})$. It is understood that the development Δ is finite.

9. Hahn has shown that if a voisinage $V(a, b)$ is defined for a class $\mathfrak{Q}$ containing at least two elements, then for every element q of $\mathfrak{Q}$ there exists a continuous function μ^q, vanishing only at q and assuming values between zero and one. Dr. Chittenden modifies the existence proof of Hahn and secures a type of uniformity among the functions μ^q. Defining (a, b) as the least upper bound of $|\mu^q(a) - \mu^q(b)|$ for all elements q of $\mathfrak{Q}$, it appears that (a, b) is an écart and is defined for every pair of elements of $\mathfrak{Q}$. By means of the uniformity mentioned it is shown that $L_n(a_n, a) = 0$ implies $L_nV(a_n, a) = 0$ and that $L_nV(a_n, a) = 0$ implies $L_n(a_n, a) = 0$. Hence voisinage is equivalent to écart. This result was anticipated by Fréchet on the ground that no theorem was known for an écart which had not been established for a voisinage. This paper will be offered to the *Transactions* for publication.

10. In a previous paper (read before the Society at the recent meeting in Columbus) Professor Carmichael laid the foundations of a general theory of the series

$$\Omega(x) = \sum_{n=0}^{\infty} c_n g(x + n),$$

where $g(x)$ is a function having, in a sector V, the Poincaré asymptotic representation

$$g(x) \sim x^{P(x)} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right),$$

$P(x)$ and $Q(x)$ being polynomials in x. In the present paper he shows that a function $\Omega(x)$ defined by the foregoing series satisfies the relations

$$\lim \{g(x + s)\}^{-1}\{\Omega(x) - \sum_{n=0}^{s} c_n g(x + s)\} = 0 \qquad (s = 0, 1, 2, \cdots),$$

provided that x approaches infinity in an appropriately deter-

mined sector $\bar{V}$ lying in V. Guided by this result he introduces a generalization of Poincaré's notion of asymptotic representation and discusses the character of this generalized representation, especially in relation to that of Poincaré.

11. In celestial mechanics, series occur very frequently which contain denominators having factors of the form $(i - j\gamma)$ where γ is an irrational number; i and j are integers which take all possible values. The study of series in which this phenomenon occurs has led Professor MacMillan to the following theorem:

If γ is a positive number, and if p_n/n is a rational fraction such that $| p_n - n\gamma | \leqq \frac{1}{2}$, and if

$$G_n = \Big(\prod_{k=1}^{n} | p_k - k\gamma |\Big)^{1/n}$$

is the geometric mean of the first n of the quantities $| p_k - k\gamma |$, then the limit of G_n, as n increases without limit, is zero if γ is rational, and is equal to $1/2e$, where $e = 2.71828 \cdots$ is the Naperian base, if γ is an irrational number which satisfies the condition that

$$a_{n+1} \leqq Mq_n(q_n + 1) \cdots (q_n + s),$$

where γ expressed as a continued fraction is

$$\gamma = a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +},$$

where q_n is the denominator of the nth principal convergent, and s is any assigned positive integer independent of n. If γ is an irrational number which does not satisfy this condition then G_n for large values of n oscillates between zero and $1/2e$.

This paper will appear in the *American Journal of Mathematics.*

12. In two former papers Professor MacMillan has shown that certain differential equations can be reduced by a linear-transcendental substitution to algebraic forms. The first of these papers dealt with a "general case," the second with equations of the first order.

The exceptions to the general case include most of the equations of dynamics, and therefore these equations require

further study. The present investigation takes up differential equations of the second order

$$\frac{dx_i}{dt} = \alpha_i x_i + \text{terms of higher degree} \quad (i = 1, 2),$$

where all of the coefficients of the right members are constants, and where the α_i considered as points in the complex plane lie on a straight line through the origin and on opposite sides of the origin. If these conditions on the α_i are not satisfied the equations belong to the general case.

If the ratio α_2/α_1 is an irrational number of a certain type (which includes all irrational algebraic numbers) then the differential equations can be reduced by a linear-transcendental substitution to its linear terms, and the substitution is convergent if the differential equations are convergent.

If the ratio α_2/α_1 is rational there is no essential restriction in supposing that $\alpha_1 = +1$, $\alpha_2 = -1$. In this case the differential equations cannot, in general, be reduced to their linear terms. They can however be reduced to a set of differential equations which are algebraic, and which are easily integrable. The convergence of the substitution is assured if the equations are canonical

$$x_1' = \frac{\partial H}{\partial x_2}, \qquad x_2' = -\frac{\partial H}{\partial x_1},$$

where H is a convergent power series in x_1 and x_2. In certain other cases, however, the substitution may diverge even though the differential equations converge. The paper will be offered to the *Transactions*.

13. Professor Fort's paper considers the development of an arbitrary function $f(x)$ in infinite series of solutions of

$$(1) \qquad \frac{d}{dx}[k(x)y'(x)] - [\lambda^2 g(x) + l(x)]\, y = 0,$$

satisfying the conditions (2) $y(0) = y(2\pi)$, $y'(0) = y'(2\pi)$; and secondly in terms of solutions satisfying the conditions (3) $y(0) = -y(2\pi)$, $y'(0) = -y'(2\pi)$. The principal results are given in the following theorem.

Let $(k)x$ and $g(x)$ be positive at all points and have second derivatives integrable from 0 to 2π. Let $l(x)$ have a first

derivative integrable from 0 to 2π and let $f(x)$ itself be integrable from 0 to 2π. Let moreover $g(0)k(0) = g(2\pi)k(2\pi)$ and $[g(0)k(0)]' = [g(2\pi)k(2\pi)]'$. Then the development corresponding to $f(x)$ in terms of solutions of (1) satisfying (2) (more accurately described in the paper) converges at any particular point of (4) $0 \leqq x \leqq 2\pi$, when and only when the Fourier series for $f(x)$ converges at that point and to the same value. It converges uniformly over the whole interval or any subinterval when and only when this is true of the Fourier series and diverges to ∞ or $-\infty$ at any particular point when and only when this is true of the Fourier series. Moreover it is summable by the method of the arithmetic mean when and only when the same thing is true of the Fourier series and to the same value.

Secondly: The development in terms of the solutions of (1) satisfying (3) bears the same relation to the development of $f(x)$ in terms of $\sin [(2n + 1)/2]x$ and $\cos [(2n + 1)/2]x$, $n = 0, 1, 2, \cdots$ as is borne by the series just discussed to the Fourier series.

14. In this paper Dr. Borger shows that if the two functions $U(x, y)$, $V(x, y)$ satisfy the following conditions in a region R:

(1) U and V are continuous in (x, y),

(2) U_x', V_x', U_y', V_y' exist and are finite,

(3) $U_x' = V_y'$; $U_y' = -V_x'$,

(4) U and V possess proper total differentials;

then U and V are analytic functions of the real variables x, y.

An immediate corollary is:

If $w = f(z) = U(x, y) + iV(x, y)$ possess a finite derivative at every point of a region R:

(*a*) This derivative is a continuous function of z.

(*b*) All the derivatives of w exist.

(*c*) w can be represented by a power series in z.

This paper will be offered to the BULLETIN for publication.

15. In this paper Professor Burgess shows that there exists a very simple relation between the exponents of the elementary divisors of the characteristic matrix $\lambda I + B$ which are connected with the linear factor $\lambda - \alpha$ of the characteristic equation $|\lambda I + B| = 0$, and the ranks of the matrices $(\alpha I + B)^n$ for $n = 1, 2, 3$, etc. In fact these ranks determine the elementary divisors and conversely.

It is also shown that the method is easily applied to any matrix of the form $\lambda A + B$ whose determinant does not vanish identically in λ. The paper will be published in the *Annals of Mathematics.*

16. In the projective theory of plane nets, as developed by Wilczynski, the members $y_1(u, v)$, $y_2(u, v)$, $y_3(u, v)$ of any fundamental system of solutions of a completely integrable system of partial differential equations of the form

$$(1)\qquad \begin{aligned} y_{uu} &= ay_u + by_v + cy, \\ y_{uv} &= a'y_u + b'y_v + c'y, \\ y_{vv} &= a''y_u + b''y_v + c''y, \end{aligned}$$

are interpreted as the homogeneous coordinates of a point P_y in a plane, defining a non-degenerate net of plane curves. The invariants and covariants of the system (1) under the transformations $y = \lambda(u, v)\bar{y}$ and $\bar{u} = U(u)$, $\bar{v} = V(v)$ may be interpreted geometrically by certain projective properties of the net. The two covariants $\rho = y_u - b'y$, $\sigma = y_v - a'y$ give rise to the homogeneous coordinates of two points P_ρ and P_σ when we substitute successively for y the values y_1, y_2, y_3. As u and v vary P_σ and P_ρ describe nets called the first and minus first Laplacian transforms of the original net. The minus first Laplacian transform of the σ-net and the first Laplacian transform of the ρ-net both coincide with the original y-net.

In this paper Dr. Hassler considers classes of nets such that the first Laplacian transform of the σ-net is the ρ-net and the minus first Laplacian transform of the ρ-net is the σ-net. Such nets are periodic of period 3 under the Laplacian transformation. He establishes necessary and sufficient conditions for such periodicity, determines the form of the coefficients of system (1) for such a net, computes its invariants and covariants, and studies the osculating conics of the curves of the original net and of its Laplacian transforms. The determination of a net of this kind by certain boundary conditions is considered and the following theorem is proved:

Choose an arbitrary triangle LMN. Through the point L pass two non-rectilinear but otherwise arbitrary analytic curves C and C', tangent to LN and LM, respectively, at L. Through each of the points M and N pass another such arbi-

trary curve, C_1' through M and C_{-1} through N, tangent to MN at M and N, respectively. Select any conic M_1 tangent to LM at M and a second conic N_{-1} tangent to LN at N. There exists one and only one net which is periodic of period 3 under the Laplacian transformation, which contains the given curves C and C', which has C_{-1} corresponding to C and C_1' corresponding to C' as curves of the minus first and first Laplacian transforms respectively, and which, moreover, has M_1 as the osculating conic of the curve of the first Laplacian transform corresponding to C and N_{-1} as the osculating conic of the curve of the minus first Laplacian transform corresponding to C'.

This paper will appear in the *Palermo Rendiconti*.

17. In this paper, Professor Hathaway shows, from the properties of gamma coefficients defined by

$$(A\alpha B\beta) = (A\alpha + B\beta)\Gamma(\alpha + \beta)/\Gamma(\alpha + 1)\Gamma(\beta + 1), \text{ etc.,}$$

that the function of weight n in the coefficients of the equation

$$x^m = a_1x^{m-1} + \cdots + a_m$$

defined by

$$F(n) = \Sigma(A_1\alpha_1A_2\alpha_2 \cdots A_n\alpha_n)a_1^{\alpha_1}a_2^{\alpha_2} \cdots a_n^{\alpha_n}$$

(the summation being for all integral solutions of

$$\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n)$$

is the unique solution of the difference equation

$$F(n) = a_1F(n-1) + \cdots + a_{n-1}F(1) + A_na_n.$$

When $n' > m$, $a_{n'} = 0$, in the above values of $F(n)$.

Hence, the problem of expanding any symmetric function of the roots is the problem of determining the coefficient A_n in its difference equation of the above type. Thus, for $\pi(n)$, the homogeneous products, $A_n = 1$, i. e., the numerical coefficient of any term in $\pi(n)$ is the multinomial coefficient of its exponents $(1\alpha_1 1\alpha_2 \cdots 1\alpha_n)$.

Also, for the homogeneous products, one root in a term, $\pi_1(n) = s_n$, by Newton's formulas, $A_n = n$. The resultant coefficient of any term $(1\alpha_1 2\alpha_2 \cdots n\alpha_n)$ is identical with Waring's.

To determine A_n for the homogeneous products, k roots

at a time, $\pi_k(n)$, we have only to consider the equation $x^n = 1$, and find $A_n = (-1)^{k+1}[1(n-k)1k]$, a binomial coefficient.

There is a remarkable relation between the homogeneous products of all weights, and those of weight n and all orders k. Thus, expanding $F(n)$ in terms of the arbitrary quantities A, we obtain

$$F(n) = A_1 a_1 \pi_{n-1} + A_2 a_2 \pi_{n-2} + \cdots + A_n a_n.$$

Thus we have the table,

	$a_1\pi_{n-1}$	$a_2\pi_{n-2}$	$a_3\pi_{n-3}$	$a_4\pi_{n-4}$	$\cdots$
$\pi(n) =$	1	1	1	1	$\cdots$
$\pi_1(n) =$	1	2	3	4	$\cdots$
$-\pi_2(n) =$		1	3	6	$\cdots$
$\pi_3(n) =$			1	4	$\cdots$
$-\pi_4(n) =$				1	$\cdots$
$\cdots$					

The relation may also be written as an identity in an arbitrary variable x,

$$\Sigma(-x)^k \pi_k(n) + \Sigma(1+x)^k a_k \pi(n-k) = \pi(n) \qquad (k = 1, 2, \cdots, n),$$

and by substituting $-1-x$ for x we see that the relation is symmetrical.

18. Cauchy's integral theorem for functions of a complex variable has often been employed to advantage for the summing of infinite series, such applications belonging to the subject commonly known as "the calculus of residues." After pointing out the limitations naturally present in using this method of summation, Professor Ford's paper proceeds to outline a way by which the fundamental difficulty may be avoided in many cases, and an actual formula for sum is obtained which has a wide range of applicability. The paper will be offered for publication to the BULLETIN.

19. The term independent generators of an abelian group has two distinct meanings. According to one of these meanings the set of operators $s_1, s_2, \cdots, s_\lambda$ is said to be a set of independent generators of the group G provided these operators generate G but no $\lambda - 1$ of them generate G. According to the other meaning such a set is called a set of independent generators of G if and only if the group generated by every $\lambda - 1$ of them has only identity in common with the group generated by the remaining one. The value of λ is the same in both cases. When the term independent generator is used with the former meaning it is said to be used in the general sense and when it is used with the latter meaning it is said to be used in the restricted sense. In the general sense the operators which cannot be used as independent generators constitute a subgroup known as the ϕ-subgroup. In the restricted sense these operators do not always constitute a subgroup.

Professor Miller's results relate to the use of independent generators in the restricted sense. If the order of an abelian group G is p^m, p being a prime number, and if we form the quotient group of G with respect to its subgroup H composed of the pth power of each of its operators, the order of this quotient group being p^λ, then λ is the number of the independent generators of G. The number of operators of G which can be used as independent generators of lowest order is equal to the number of operators of lowest order in the co-sets of G with respect to H. The totality of the operators which can be used as independent generators of other orders may be determined similarly.

20. The continuity axioms of Cantor and Weierstrass are essentially assumptions about the existence of limits. Professor Frizell submits a set of postulates based on ideas of order, from which it is easy to deduce both Cantor's assumption and the principle of Archimedes, and therefore also the postulates of Weierstrass and Dedekind. This paper will be offered to the BULLETIN for publication.

21. Professor Williams's paper appears in the present number of the BULLETIN.

22. A real function $f(x)$ uniquely defined in the interval (a, b)

is said to be "convex," if for every two values x_1 and x_2 in the interval (a, b) the inequality

$$f\left(\frac{x_1 + x_2}{2}\right) \leqq \frac{f(x_1) + f(x_2)}{2}$$

holds.* From a paper of Bernstein and Doetsch,† according to which a convex function is either continuous (except possibly at the ends of the interval) or totally discontinuous, it appears that discontinuous convex functions are of relatively minor importance. Let then the continuity of $f(x)$ be assumed henceforth. Jensen has shown‡ that the derivative of $f(x)$ may be non-existent in an everywhere dense set of points, but the right-hand derivative (together with the left-hand derivative) always exists. Professor Blumberg proves (among other things): (*a*) The right-hand derivative of $f(x)$ is a monotone increasing function; conversely, if the right-hand derivative of a continuous function $g(x)$ exists everywhere and is a monotone increasing function, then $g(x)$ is convex. (*b*) The right-hand derivative of $f(x)$ is continuous on the right at every point of (a, b). (*c*) The points of (a, b) at which the derivative of $f(x)$ is non-existent form at most a denumerable set (conversely, given a denumerable set S, a convex function $f(x)$ may be constructed, such that the derivative of $f(x)$ is existent at every point not in S and non-existent at every point in S). (*d*) Every continuous convex function is the indefinite integral of a monotone increasing function; conversely, the indefinite integral of a monotone increasing function is convex (and continuous).

The proofs are extremely simple. Moreover, the proofs of most of the theorems (on continuous convex functions) contained in the two papers quoted above are simplified and rendered nearly intuitionally evident; the Cauchy artifice,§ which is at the basis of the proofs of Jensen, Bernstein and Doetsch, may be altogether dispensed with.

23. The theorem in question is as follows: || The number

$$f(x) = \Sigma_{r=0}^{r=\infty} \alpha_r x^r / a^{c^r}$$

---

* See Jensen, *Acta Math.*, vol. 30, p. 176.
† *Math. Ann.*, 1915, p. 514.
‡ L. c.
§ See Jensen, l. c., p. 175.
|| See BULLETIN, Mar., 1915, p. 285.

is transcendental for every rational (real) x ($\neq 0$), if a and c are integers $\geqq 2$ and if (1) $|\alpha_r| < M^r$, M arbitrary but fixed, and (2) only a finite number of the α's vanish. Professor Blumberg shows by means of a modification of Dr. Kempner's method that the condition (2) may be substantially altogether dropped and replaced by the restriction (evidently to be demanded) that $f(x)$ shall not break off after a finite number of terms. Condition (1) may be replaced by the milder condition $|\alpha_r| < M^{r^k}$, k being like M arbitrary but fixed.

25. In this paper Mr. Dantzig considers a continuous plane n to 1 transformation T. Let C be an arbitrary curve through a point P and having t for tangent at P and let C_1P_1 be the images of C and P, and t_1 the tangent to C_1 at P_1; t and t_1 will meet in a point τ. The locus of the point τ for the totality of the curves C through P is a conic, which the author calls the indicatrix of T for the point P. There exists for each point in the plane such an indicatrix I, and the totality of the indicatrices I form a two-parameter system of conics. It is further shown that the behavior of T at a point P is characterized by the indicatrix at P. In particular if the totality of the indicatrices form a system of circles or equilateral hyperbolas, the transformation is conformal.

The case of the degeneracy of the indicatrix is further taken up and the intimate relation of the indicatrix to the invariant elements of T is brought out. It is shown that in the case of the identical degeneracy of the indicatrix either T is a pseudo transformation, or there exists an infinity of invariant straight lines enveloping an absolutely invariant curve.

The value of the method for the classification of plane transformations is discussed. Applied to collineations the method furnishes a purely geometrical basis for projective geometry independent of any consideration of anharmonic ratios, elliptic involutions or polar reciprocity.

The author applies this method to a great number of problems in geometry, particularly to the proof of Poincaré's last theorem, to the problem of closure and the theory of unicursal plane curves.

26. By means of a remarkably simple generalization of the concept limit of a function Professor E. H. Moore has recently found it possible to define the integration process J in a new

theory of linear integral equations in general analysis, a general theory having as its guiding instance a body of theorems, due to Hilbert and developed more directly by Hellinger, concerning limited linear, quadratic and hermitian forms in infinitely many numerical variables. In the present paper, which will be offered to the *Transactions*, Professor Moore gives definitions of the new limit and of various allied concepts, proves a number of fundamental propositions involving these concepts, and points out the sense in which various classical limits are instances of the new limit. The definition is as follows: Consider a class $\mathfrak{Q}$ of elements q and a binary relation K on $\mathfrak{Q}$ which is reflexive, transitive and of such a nature that for every two elements q_1, q_2 there is an element q_{12} of such a nature that $q_{12}\,\mathrm{K}\,q_1$ and $q_{12}\,\mathrm{K}\,q_2$. (Examples: I. $\mathfrak{Q}$ is the class of positive integers q; K is the binary relation $\geqq$ on $\mathfrak{Q}$. II. $\mathfrak{Q}$ is the class of all finite subclasses q of a class $\mathfrak{P}$ of elements p; K is the binary relation $\supset$, inclusion, on $\mathfrak{Q}$, viz., for two classes q_1, q_2, $q_1 \supset q_2$ denotes that every element p of q_2 is an element of q_1.) Then, with respect to this relation K, a numerically valued function φ or $\varphi(q)$ on $\mathfrak{Q}$ has the number a as limit in case for every positive number e there exists an element q_e depending on e of such a nature that for every element $q\,\mathrm{K}\,q_e$ it is true that $|\varphi(q) - a| \leqq e$. (Accordingly, in example I, the number a is the limit in the classical sense of the numerical sequence $\varphi(1), \varphi(2), \cdots, \varphi(q), \cdots$. The limit of example II is the limit used in the definition of the integration process J; it is to be noted that this limit is of general reference in the sense that it has reference to no metric or other features of the class $\mathfrak{P}$ which is accordingly a (truly) general class.)

27. The differentiating machine designed by Mr. Elmendorf plots the differential curve of any given curve, and is primarily applicable in drawing the rate curve for any empirical curve. A silver reflector is mounted at the end of a bar and at right angles to it so that when the mirror is set upon the curve and turned until the image and the actual curve form a continuous line, the bar is tangent to the curve at the point in question. The lengths cut off by the bar on a vertical erected at one end of a horizontal link of constant length which is so arranged that the other end is over the center of the mirror, are plotted as ordinates of the desired differential curve. An

account of the machine was published in the *Scientific American Supplement* for Feb. 12, 1916.

28. A cubic which has eight consecutive points in common with a curve C at a point P is called an eight-pointic cubic of P. One of the ∞^1 eight-pointic cubics has a double point at P and is known as the eight-pointic nodal cubic. The double point tangents are, in general, distinct and form, with the inflectional line of the cubic, a triangle, which is called the canonical triangle.

The anharmonic curve which has eight consecutive points in common with C at the point P is called the osculating anharmonic curve. Associated with an anharmonic curve is, in general, a triangle, called the invariant triangle, which has the following property. Suppose any tangent is drawn to the curve. The anharmonic ratio of the point of tangency and the three points of intersection of the tangent with the sides of the invariant triangle is constant for all points on the curve.

Associated with every point P on the curve C there are six points and six lines which form the vertices and sides of the canonical triangle and the invariant triangle of the osculating anharmonic. As P moves along the curve C the six points will describe certain loci and the six lines will envelope certain curves. The object of Mr. Harding's paper is to study the character of these loci. The paper will be offered for publication to the *Giornale di Matematiche.*

29. Professor Wilczynski has shown that the theory of nonruled analytic surfaces may be based on a completely integrable system of partial differential equations of the form

$$\frac{\partial^2 y}{\partial u^2} + 2b\frac{\partial y}{\partial v} + fy = 0,$$

$$\frac{\partial^2 y}{\partial v^2} + 2a'\frac{\partial y}{\partial u} + gy = 0.$$

In the present paper, Miss Sperry associates with each point of the surface a line through the point but not in the tangent plane of the point. In this way a congruence L is determined. There are two one-parameter families of curves on the surface along which the lines of L generate developables. We have called them the congruential torsal curves. On every surface

there is a two-parameter family of curves having the property that the osculating planes of all the curves of the family passing through a point have in common the line of L through that point. Their differential equation is

$$\frac{d^2u}{dt^2}\frac{dv}{dt} - \frac{d^2v}{dt^2}\frac{du}{dt} + 2\left\{b\left(\frac{du}{dt}\right)^3 - a'\left(\frac{dv}{dt}\right)^3\right\} + 2\left\{p_1\left(\frac{du}{dt}\right)^2\frac{dv}{dt} + p_2\frac{du}{dt}\left(\frac{dv}{dt}\right)^2\right\} = 0,$$

where p_1 and p_2 depend upon the choice of L.

Conversely the integral curves of an equation of this type have the property that the osculating planes of all the curves of the family which pass through a point form a pencil. These curves we have called the congruential union curves. If L is the congruence of surface normals the torsal curves are lines of curvature and the union curves are geodesics. If L is the congruence of directrices of the second kind the torsal curves are directrix curves.

A necessary and sufficient condition that the union curves be plane is that they be torsal curves, a generalization of a well known theorem concerning geodesics and lines of curvature.

30. With every point P of a non-ruled surface S Professor Wilczynski has associated a pair of straight lines; one of these lines lies in the tangent plane and is called the directrix of the first kind, while the other pierces the surface at P and is called the directrix of the second kind. The two families of developables of the two congruences thus associated with S are determined by one and the same net of curves, the directrix curves on S.

In this paper Dr. Yeaton obtains the conditions under which the focal sheets of the directrix congruence of the second kind degenerate into curves. Limiting the discussion to surfaces whose directrix curves form a conjugate net, he shows that if this net is not degenerate, the congruence in question may be linear. The surfaces thus determined are projectively equivalent to the surface

$$z = \frac{1}{4}xy\left(\log_e\frac{x}{y}\right)^2$$

On any surface the two families of asymptotic curves are projectively equivalent, each lies on a quadric and is identically self-dual; the directrix curves are plane curves and one of the two families consists of conics. A non-degenerate quadric and a straight line, which is not a ruling of the quadric, constitute the focal surface of the directrix congruence of the first kind. The finite equations of the various associated loci are obtained.

ARNOLD DRESDEN,
Secretary of the Section.

NOTE ON FUNCTIONS OF SEVERAL COMPLEX VARIABLES.

BY PROFESSOR WILLIAM F. OSGOOD.

(Read before the American Mathematical Society, April 29, 1916.)

THE object of the present note is at once to extend the scope of a fundamental theorem of the theory of analytic functions of several complex variables and to simplify its proof.*

Definition.—Let S be the cylindrical region $(S_1, \cdots, S_n)$,

$$S_k: \qquad |z_k| < r_k \qquad (k = 1, \cdots, n);$$

let Σ be the region $(\Sigma_1, \cdots, \Sigma_n)$,

$$\Sigma_j: \qquad |z_j| < h_j < r_j \qquad (j = 1, 2);$$

$$\Sigma_k: \qquad |z_k| < r_k \qquad (k = 3, \cdots, n);$$

and let T be the region whose points are interior to S, but exterior to Σ:

$$T = S - \Sigma.\dagger$$

THEOREM. *Let $f(z_1, \cdots, z_n)$ be analytic throughout the region T. Then $f(z_1, \cdots, z_n)$ admits analytic continuation throughout S.*

---

* The theorem was given by Kistler, "Ueber Funktionen von mehreren komplexen Veränderlichen," § 7, Basel, 1905, for the case that the excepted points lie on a finite number of analytic manifolds, each of $n - 2$ complex dimensions, and was proven by means of n-fold integrals.

† This symbolic form is suggestive, but not quite accurate, since it would assign to T certain of its boundary points, and T consists only of interior points.

The proof is given at once by Cauchy's integral formula for functions of a single variable—for simplicity we set $n = 3$—

$$f(z_1, z_2, z_3) = \frac{1}{2\pi i}\int_C \frac{f(z_1, t_2, z_3)dt_2}{t_2 - z_2}.$$

Here C shall be a circle,

$$|t_2| = r_2', \quad h_2 < r_2' < r_2,$$

r_2' being taken as near to r_2 as one pleases. Furthermore, z_1 shall be a point of the ring

$$h_1 < |z_1| < r_1,$$

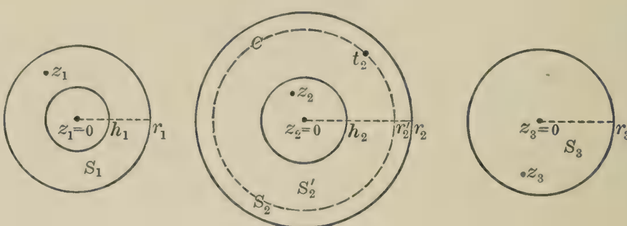


FIG. 1.

while z_3 is any point of the circle

$$|z_3| < r_3.$$

If finally z_2 is any point interior to the circle C, the hypotheses of the theorem justify the above formula.

But the integrand, for any fixed point t_2 on the circle C, is analytic throughout the whole region $S' = (S_1, S_2', S_3)$,

$$S_2': \qquad |z_2| < r_2';$$

and it is continuous when (z) lies in S' and t_2 on C. Hence the integral represents a function analytic throughout S'.

Thus $f(z_1, z_2, z_3)$ admits analytic continuation throughout S', and hence finally throughout S.

Remark. The foregoing theorem is contained in a theorem of Hartogs's.* Let a_1 be a point of the region $h_1 < |z_1| < r_1$, and let $a_3 = 0$. Then

(i) $f(z_1, z_2, z_3)$ is analytic in each point (a_1, z_2, a_3), where z_2 is any point of S_2', including the boundary;

* *Sitzungsber. der Münchener Akad.*, 36 (1906), p. 223. Cf. also the Madison Colloquium, pp. 168, 169.

(ii) $f(z_1, z_2, z_3)$ is analytic in each point (z_1, t_2, z_3), where t_2 is any point of C, and z_1, z_3 lie respectively in S_1 and S_3.

Hence $f(z_1, z_2, z_3)$ admits analytic continuation throughout S', and thus throughout S.

Hartogs's proof of the more general theorem is less simple, involving as it does n-fold integrals.

HARVARD UNIVERSITY,
April 17, 1916.

QUASI-PERIODICITY OF ASYMPTOTIC PLANE NETS.

BY DR. ALFRED L. NELSON.

(Read before the American Mathematical Society, April 21, 1916.)

1. *Introduction.*—The projective properties of plane nets of curves have been discussed by Wilczynski.* For this purpose he makes use of a certain completely integrable system of three linear homogeneous partial differential equations of the second order, namely,

$$y_{uu} = ay_u + by_v + cy,$$
$$(1)\qquad y_{uv} = a'y_u + b'y_v + c'y,$$
$$y_{vv} = a''y_u + b''y_v + c''y.$$

Three linearly independent solutions of this system, $y^{(k)}$ $(k = 1, 2, 3)$, are interpreted as the homogeneous coordinates of a point P_y which generates the plane net. The projective properties of the net are expressed in terms of the invariants of (1) under the transformations

$$(2)\qquad y = \lambda(u, v)\bar{y}; \qquad \bar{u} = U(u), \qquad \bar{v} = V(v).$$

Two of these invariants,

$$H = c' + a'b' - a_u', \quad K = c' + a'b' - b_v',$$

the so-called Laplace-Darboux invariants, are expressed entirely in terms of the middle equation, which is of the type

* Wilczynski, "One-parameter families and nets of plane curves," *Transactions Amer. Math. Society*, vol. 12 (1911), no. 4, pp. 473–510.

studied by Laplace.* A certain special case, namely, when the invariants H and K are equal, acquires especial interest in view of a theorem of Koenigs:† *The perspectives of the asymptotic curves of a surface from a fixed point on a fixed plane form a net with equal invariants. Conversely, a plane net with equal invariants may be regarded as the perspectives from a fixed point of the asymptotic curves of a surface.* We shall speak of a plane net for which $H = K$ as an *asymptotic* plane net. The present paper has as its object the discussion of the Laplace transformation, to which a large part of Wilczynski's paper is devoted, for the special case mentioned. The characteristic system of partial differential equations for the general Laplace transformed net will be computed, together with its fundamental invariants, and certain theorems concerning quasi-periodic plane nets will be deduced.

The first Laplace transform is the net generated by the covariant point $P_{y'}$, whose coordinates are $y'^{(k)} = y_v^{(k)} - a'y^{(k)}$ $(k = 1, 2, 3)$. The minus first Laplace transform is described by the covariant point $P_{y^{(-1)}}$, where $y^{(-1)} = y_u - b'y$. Each of these new nets has a first and a minus first Laplace transform, and it is readily shown that the minus first transform of the first transform, as well as the first transform of the minus first transform, is the original net. Accordingly, we have, in general, an infinite chain of nets, the Laplace suite, covariantly connected with the original net. Each net of the suite has, of course, a characteristic system of equations, and we shall call the system of the ith transform $(Y^{(i)})$ (i a positive or negative integer), and distinguish the coefficients and invariants of this transform by the subscript i.

Under the assumption of the equality of H and K, the system (1) may be put in the unique form

$$\begin{aligned} &y_{uu} = -2\frac{\delta_u}{\delta}\cdot y_u + \mathfrak{B}y_v - \mathfrak{B}\frac{\partial}{\partial v}\log\frac{\mathfrak{B}}{\delta^2}\cdot y, \\ (3)\qquad (Y)\quad &y_{uv} = Hy, \quad \left(H = \mathfrak{A}''\mathfrak{B} - \frac{\partial^2 \log \delta^2}{\partial u \partial v}\right), \\ &y_{vv} = \mathfrak{A}''y_u - 2\frac{\delta_v}{\delta}\cdot y_v - \mathfrak{A}''\frac{\partial}{\partial u}\log\frac{\mathfrak{A}''}{\delta^2}\cdot y, \end{aligned}$$

* "Recherches sur le calcul intégral aux différences partielles." Oeuvres de Laplace, t. IX, pp. 29, et seq.

† Koenigs, "Sur les réseaux plans à invariants égaux et les lignes asymptotiques." *Comptes Rendus*, vol. 114 (1892), p. 55.

which is characterized by the relations $a' = b' = 0$, and where δ is a non-vanishing function of u and v.* The integrability conditions of (Y) are obtained from the identical relations

$$(y_{uu})_v = (y_{uv})_u; \quad (y_{uv})_v = (y_{vv})_u.$$

They are given, for the general case, as equations (5) of Wilczynski's paper. Each of the other nets of the Laplace suite has the six corresponding integrability conditions, which are satisfied as a result of those of the original net. We shall, however, use the integrability conditions of the other nets of the suite as being, by their form, better adapted to the simplification of our computations, and shall refer to them as the first, second, etc., the order being understood to be the same as in the case of those of the original net, as given by Wilczynski.

2. *The General Laplace Transform.*—Let us assume that the $(k+1)$th (k a positive integer) transform is non-degenerate,† and has the following coefficients:

$$
\begin{aligned}
&a_{k+1} = \frac{\partial}{\partial u} \log \frac{\mathfrak{A}_k''!\, H_k!}{\delta^2}, & & b_{k+1} = \frac{H_k}{\mathfrak{A}_k''}, \\
&a'_{k+1} = \frac{\partial}{\partial v} \log H_k!, & & b'_{k+1} = 0, \\
&a''_{k+1} = \mathfrak{A}''_{k+1}, & & b''_{k+1} = \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{\delta^2}, \\
(4) \qquad &c_{k+1} = -\frac{H_k}{\mathfrak{A}_k''} \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}''_{k-1}!}{H_{k-1}!\, \delta^2}, \\
&c'_{k+1} = H_k, \\
&c''_{k+1} = c_k'' + \frac{\partial^2}{\partial v^2} \log \frac{\mathfrak{A}''_{k-1}!}{(H_{k-1}!)^2 \delta^2} \\
&\qquad\qquad - \frac{\partial}{\partial v} \log \mathfrak{A}_k'' \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}''_{k-1}!}{H_{k-1}!\, \delta^2},
\end{aligned}
$$

* The transformation which accomplishes this, with the help of the integrability conditions, is of the form of the first of (2), and does not alter the net.

† It is easily shown that for the $(k+1)$th transform to degenerate into a single curve it is necessary and sufficient that $\mathfrak{A}_k'' H_k = 0$ (cf. (5)), assuming that the kth transform is not degenerate. The assumption that the $(k+1)$th transform is non-degenerate carries with it, of course, the assumption of the non-degeneracy of all the transforms up to the $(k+1)$th.

where

$$
(5) \qquad \begin{aligned} H_j &= H_{j-1} - \frac{\partial^2}{\partial u \partial v} \log H_{j-1}!, \\ \mathfrak{A}_j'' &= \frac{\mathfrak{A}_{j-1}''}{H_{j-1}} \left(H_{j-1} - \frac{\partial^2}{\partial u \partial v} \log \frac{\mathfrak{A}_{j-1}''!}{\delta^2} \right), \end{aligned}
$$

and

$$
\theta_j! = \theta_0 \theta_1 \theta_2 \cdots \theta_j, \quad \theta_0 = \theta,
$$

$$
(j \text{ a positive integer, } \theta = \mathfrak{A}'' \text{ or } H).
$$

The coefficients of the first and second transforms, if these transforms are not degenerate, can readily be shown to be of the form (4).

The $(k+2)$th transform is generated by the point $P_{y^{(k+2)}}$, where

$$
(6) \qquad y^{(k+2)} = y_v^{(k+1)} - \frac{\partial}{\partial v} \log H_k! \cdot y^{(k+1)}.
$$

By differentiation of (6) and application of the equations given by (4), we find the relations

$$
(7) \qquad \begin{aligned} y_u^{(k+2)} &= H_{k+1} y^{(k+1)}, \\ y_v^{(k+2)} &= \mathfrak{A}_{k+1}'' y_u^{(k+1)} + \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k! \, \delta^2} \cdot y_v^{(k+1)} \\ &\quad + \left(c_{k+1}'' - \frac{\partial^2}{\partial v^2} \log H_k! \right) y^{(k+1)}. \end{aligned}
$$

Except when $\mathfrak{A}_{k+1}'' H_{k+1} = 0$, that is, except when the $(k+2)$th transform is degenerate, we may solve equations (6) and (7) for $y^{(k+1)}$, $y_u^{(k+1)}$, $y_v^{(k+1)}$, obtaining the expressions

$$
(8) \qquad \begin{aligned} y^{(k+1)} &= \frac{1}{H_{k+1}} y_u^{(k+2)}, \\ y_v^{(k+1)} &= y^{(k+2)} + \frac{1}{H_{k+1}} \frac{\partial}{\partial v} \log H_k! \cdot y_u^{(k+2)}, \\ y_u^{(k+1)} &= \frac{1}{H_{k+1}} \frac{\partial}{\partial u} \log \frac{\mathfrak{A}_{k+1}''! \, H_k!}{\delta^2} \cdot y_u^{(k+2)} \\ &\quad + \frac{1}{\mathfrak{A}_{k+1}''} y_v^{(k+2)} - \frac{1}{\mathfrak{A}_{k+1}''} \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k! \, \delta^2} \cdot y^{(k+2)}, \end{aligned}
$$

where the fourth integrability condition of $(Y^{(k+1)})$ has been used to obtain the given form of the coefficient of $y_u^{(k+2)}$ in the third equation of (8).

If we differentiate equations (7) again, and make use of (4), we find the following equations:

$$
\begin{aligned}
y_{uu}{}^{(k+2)} &= (H_{k+1})_u \cdot y^{(k+1)} + H_{k+1} \cdot y_u{}^{(k+1)}, \\
y_{uv}{}^{(k+2)} &= (H_{k+1})_v \cdot y^{(k+1)} + H_{k+1} \cdot y_v{}^{(k+1)}, \\
y_{vv}{}^{(k+2)} &= \Bigg(c''_{k+1} + \frac{\partial^2}{\partial v^2} \log \frac{\mathfrak{A}_k''!}{(H_k!)^2\delta^2} \\
&\quad + \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{\delta^2} \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k!\,\delta^2} \Bigg) y_v{}^{(k+1)} \\
&\quad + \mathfrak{A}''_{k+1} \frac{\partial}{\partial v} \log \frac{\mathfrak{A}''_{k+1}!}{\delta^2} \cdot y_u{}^{(k+1)} \qquad (9) \\
&\quad + \Bigg(c''_{k+1} \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!\, c''_{k+1}}{H_k!\,\delta^2} - \frac{\partial^3}{\partial v^3} \log H_k! \\
&\quad + H_k \mathfrak{A}''_{k+1} \Bigg) y^{(k+1)}.
\end{aligned}
$$

Substitution of (8) in (9) yields the characteristic system of equations of the $(k+2)$th transform, for which the following are the coefficients:

$$
\begin{aligned}
a_{k+2} &= \frac{\partial}{\partial u} \log \frac{\mathfrak{A}''_{k+1}!\, H_{k+1}!}{\delta^2}, & b_{k+2} &= \frac{H_{k+1}}{\mathfrak{A}''_{k+1}}, \\
a'_{k+2} &= \frac{\partial}{\partial v} \log H_{k+1}!, & b'_{k+2} &= 0, \\
a''_{k+2} &= \mathfrak{A}''_{k+2}, & b''_{k+2} &= \frac{\partial}{\partial v} \log \frac{\mathfrak{A}''_{k+1}!}{\delta^2}, \\
c_{k+2} &= -\frac{H_{k+1}}{\mathfrak{A}''_{k+1}} \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k!\,\delta^2}, \qquad (10) \\
c'_{k+2} &= H_{k+1}, \\
c''_{k+2} &= c''_{k+1} + \frac{\partial^2}{\partial v^2} \log \frac{\mathfrak{A}_k''!}{(H_k!)^2\delta^2} \\
&\quad - \frac{\partial}{\partial v} \log \mathfrak{A}''_{k+1} \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_k''!}{H_k!\,\delta^2},
\end{aligned}
$$

where the expression for a''_{k+2} (cf. (5)) is given by the fifth integrability condition of $(Y^{(k+1)})$. Comparison of (10) with

(4) shows that we have completed the induction proof that (4) gives the characteristic system $(Y^{(k+1)})$ for the $(k+1)$th transform (k positive).

We may compute the general minus $(k+1)$th transform in the same manner, or may more easily obtain the result by applying the substitution

$$\begin{pmatrix} \mathfrak{A}_j'' & u & c_k'' \\ \mathfrak{B}_{-j} & v & c_k \end{pmatrix} \quad (j = k-1, k, k+1)$$

to the equations given by (4). The coefficients of the minus $(k+1)$th (k positive) turn out to be

$$\begin{aligned} (11) \quad & a_{-(k+1)} = \frac{\partial}{\partial u} \log \frac{\mathfrak{B}_{-k}!}{\delta^2}, \quad b_{-(k+1)} = \mathfrak{B}_{-(k+1)}, \\ & c_{-(k+1)} = c_{-k} + \frac{\partial^2}{\partial u^2} \log \frac{\mathfrak{B}_{-(k-1)}!}{(H_{k-1}!)^2\delta^2} \\ & \qquad\qquad - \frac{\partial}{\partial u} \log \mathfrak{B}_{-k} \cdot \frac{\partial}{\partial u} \log \frac{\mathfrak{B}_{-(k-1)}!}{H_{k-1}!\,\delta^2}, \\ & a'_{-(k+1)} = 0, \qquad b'_{-(k+1)} = \frac{\partial}{\partial u} \log H_k!, \\ & c'_{-(k+1)} = H_k, \\ & a''_{-(k+1)} = \frac{H_k}{\mathfrak{B}_{-k}}, \quad b''_{-(k+1)} = \frac{\partial}{\partial v} \log \frac{\mathfrak{B}_{-k}!\,H_k!}{\delta^2}, \\ & c''_{-(k+1)} = -\frac{H_k \partial}{\mathfrak{B}_{-k} \partial u} \log \frac{\mathfrak{B}_{-(k-1)}!}{H_{k-1}!\,\delta^2}, \end{aligned}$$

where

$$(12) \quad \mathfrak{B}_{-j} = \frac{\mathfrak{B}_{-(j-1)}}{H_{j-1}} \left(H_{j-1} - \frac{\partial^2}{\partial u \partial v} \log \frac{\mathfrak{B}_{-(j-1)}!}{\delta^2} \right)$$

(j any positive integer),

and

$$\mathfrak{B}_{-j}! = \mathfrak{B}_0 \mathfrak{B}_{-1} \mathfrak{B}_{-2} \cdots \mathfrak{B}_{-j}.$$

We have already mentioned the fact that for the $(k+2)$th transform to degenerate, while the $(k+1)$th transform is non-degenerate, it is necessary and sufficient that $\mathfrak{A}_{k+1} = 0$, or $H_{k+1} = 0$. Reference to the third equation of the system determined by (4) shows that *the first of these conditions is*

necessary and sufficient for the curves $u = const.$ of the $(k+1)$th transform to be straight lines. The first equation of (4) shows that *the curves $v = const.$ of the $(k+1)$th transform cannot be straight lines,* since the condition for this, $b_{k+1} = 0$, would make the $(k+1)$th transform degenerate. Similarly, *degeneracy of the minus $(k+2)$th transform is equivalent to $\mathfrak{B}_{-(k+1)} = 0$, or $H_{k+1} = 0$. The first of these conditions is necessary and sufficient for the curves $v = const.$ of the minus $(k+1)$th transform to be straight lines. The curves $u = const.$ of this transform cannot be straight lines.*

3. *Summary of Fundamental Invariants.*—In the paper referred to, Wilczynski has proved the theorem: *If the invariants $\mathfrak{B} = b$, $\mathfrak{C} = c + a'b + ab' - b'^2 - b_u'$, $\mathfrak{A}' = \frac{2}{3}a' - \frac{1}{3}b'' + \frac{1}{6}(a_v''/a'')$, $\mathfrak{B}' = \frac{2}{3}b' - \frac{1}{3}a + \frac{1}{6}(b_u/b)$, $\mathfrak{C}' = c' + a'b' - \frac{1}{3}(a_v + b_v')$, $\mathfrak{A}'' = a''$, $\mathfrak{C}'' = c'' + a''b' + a'b'' - a'^2 - a_v'$, of a net are given as functions of u and v, subject to the integrability conditions, the net is determined, except for a projective transformation.** Omitting the details of computation, these fundamental invariants, together with the invariants H and K, for the $(k+1)$th transform have the following expressions:

$$
\begin{aligned}
&\mathfrak{B}_{k+1} = \frac{H_k}{\mathfrak{A}_k''},\\
&\mathfrak{C}_{k+1} = -\frac{H_k}{\mathfrak{A}_k''}\cdot\frac{\partial}{\partial v}\log\frac{\mathfrak{A}_{k-1}''!}{(H_{k-1}!)^2H_k\delta^2},\\
&\mathfrak{A}_{k+1}' = \frac{1}{6}\frac{\partial}{\partial v}\log\frac{(H_k!)^4\delta^4\mathfrak{A}_{k+1}''}{(\mathfrak{A}_k''!)^2},\\
(13)\qquad &\mathfrak{B}_{k+1}' = \frac{1}{6}\frac{\partial}{\partial u}\log\frac{H_k\delta^4}{(\mathfrak{A}_k''!)^2\mathfrak{A}_k''(H_k!)^2},\\
&\mathfrak{C}_{k+1}' = H_k - \frac{1}{3}\frac{\partial^2}{\partial u\partial v}\log\frac{\mathfrak{A}_k''!\,H_k!}{\delta^2},\\
&\mathfrak{A}_{k+1}'' = \mathfrak{A}_{k+1}'',\\
&\mathfrak{C}_{k+1}'' = -\mathfrak{A}_{k+1}''\frac{\partial}{\partial u}\log\frac{\mathfrak{A}_{k+1}''!\,H_k!}{\delta^2},\\
&H_{k+1} = H_{k+1}, \qquad K_{k+1} = H_k.
\end{aligned}
$$

* P. 485 of Wilczynski's paper.

The corresponding invariants of the minus $(k+1)$th transform are

$$
\begin{aligned}
\mathfrak{B}_{-(k+1)} &= \mathfrak{B}_{-(k+1)},\\
\mathfrak{C}_{-(k+1)} &= -\,\mathfrak{B}_{-(k+1)}\frac{\partial}{\partial v}\log\frac{\mathfrak{B}_{-(k+1)}!\,H_k!}{\delta^2},\\
\mathfrak{A}'_{-(k+1)} &= \frac{1}{6}\frac{\partial}{\partial v}\log\frac{H_k\delta^4}{(\mathfrak{B}_{-k}!)^2\mathfrak{B}_{-k}(H_k!)^2},\\
\mathfrak{B}'_{-(k+1)} &= \frac{1}{6}\frac{\partial}{\partial u}\log\frac{(H_k!)^4\mathfrak{B}_{-(k+1)}\delta^4}{(\mathfrak{B}_{-k}!)^2},\\
\mathfrak{C}'_{-(k+1)} &= H_k-\frac{1}{3}\frac{\partial^2}{\partial u\partial v}\log\frac{\mathfrak{B}_{-k}!\,H_k!}{\delta^2},\\
\mathfrak{A}''_{-(k+1)} &= \frac{H_k}{\mathfrak{B}_{-k}},\\
\mathfrak{C}''_{-(k+1)} &= -\frac{H_k}{\mathfrak{B}_{-k}}\cdot\frac{\partial}{\partial u}\log\frac{\mathfrak{B}_{-(k-1)}!\,H_k}{(H_k!)^2\delta^2},\\
H_{-(k+1)} &= H_k, \qquad K_{-(k+1)} = H_{k+1}.
\end{aligned}
\tag{14}
$$

4. *Quasi-periodic Nets.*—In case the ith transform is projectively equivalent to the $(i+j)$th transform (j a positive integer, i a positive or negative integer), the net is said to be quasi-periodic, of period j. Let us assume that $\mathfrak{A}_k''$, $\mathfrak{B}_{-k}$ and H_k are different from zero, so that the $(k+1)$th and minus $(k+1)$th transforms are non-degenerate. A necessary and sufficient condition for a quasi-periodic net, of period $2(k+1)$, is that the fundamental invariants of $(Y^{(k+1)})$ be equal to the corresponding invariants of $(Y^{-(k+1)})$. If such a period exists, however, the invariants H of the two transforms must also be equal. By reference to (5), we see that this condition is

$$
\frac{\partial^2}{\partial u\partial v}\log H_k! = 0,
\tag{15}
$$

so that by a transformation of the form $\bar{u} = U(u)$, $\bar{v} = V(v)$ we may make $H_k! = 1$. But (15) also makes the $(k+1)$th and minus $(k+1)$th transforms asymptotic, so that we may take advantage of the following theorem: *If the coefficients a, b, a'', b'', of the form* (3) *of the differential equations of an*

*asymptotic plane net are given as functions of u and v, subject to the integrability conditions, they determine the net except for a projective transformation.** For the case of quasi-period $2(k+1)$, the transformation which makes $H_k! = 1$ also puts the differential equations of the $(k+1)$th and the minus $(k+1)$th transforms, namely, those given by (10) and (11), in the form (3). Hence we obtain the further conditions desired by equating the coefficients a_{k+1}, b_{k+1}, a''_{k+1}, b''_{k+1}, to the corresponding coefficients of the minus $(k+1)$th transform. Remembering that $H_k! = 1$, these conditions are the following:

$$(16)\qquad \begin{aligned} \frac{\partial}{\partial u}\log\frac{\mathfrak{A}_k''!}{\delta^2} &= \frac{\partial}{\partial u}\log\frac{\mathfrak{B}_{-k}!}{\delta^2}, \\ \frac{H_k}{\mathfrak{A}_k''} &= \mathfrak{B}_{-(k+1)}, \\ \mathfrak{A}''_{k+1} &= \frac{H_k}{\mathfrak{B}_{-k}}, \\ \frac{\partial}{\partial v}\log\frac{\mathfrak{A}_k''!}{\delta^2} &= \frac{\partial}{\partial v}\log\frac{\mathfrak{B}_{-k}!}{\delta^2}. \end{aligned}$$

The first and fourth of these equations imply $\mathfrak{A}_k''!/\mathfrak{B}_{-k}! =$ constant, and we may make this constant equal to unity by a suitable transformation of the independent variables, without violating the condition $H_k! = 1$. The second and third equations of (16) are now equivalent, in view of (5) and (12), so that we have the theorem: *If an asymptotic plane net, for which the $(k+1)$th and minus $(k+1)$th transforms are not degenerate, is quasi-periodic, of period $2(k+1)$, its differential equations may be so written that*

$$(17)\qquad H_k! = 1, \quad \mathfrak{A}_k''! = \mathfrak{B}_{-k}!, \quad \mathfrak{A}_{k+1}'' = H_k/\mathfrak{B}_{-k}.$$

Conversely, any asymptotic plane net, whose $(k+1)$th and minus $(k+1)$th transforms are non-degenerate, and for which equations (17) hold, is quasi-periodic, of period $2(k+1)$.

In order to discuss the case of odd quasi-periods, we equate the corresponding fundamental invariants of $(Y^{(-k)})$ and $(Y^{(k+1)})$, to obtain the conditions for a quasi-period $2k+1$. The conditions are the following:

---

* The truth of this theorem is easily seen by reference to (3).

$$\mathfrak{B}_{-k} = \frac{H_k}{\mathfrak{A}_k''},$$

$$- \mathfrak{B}_{-k} \frac{\partial}{\partial v} \log \frac{\mathfrak{B}_{-k}!\, H_{k-1}!}{\delta^2} = - \frac{H_k}{\mathfrak{A}_k''} \cdot \frac{\partial}{\partial v} \log \frac{\mathfrak{A}_{k-1}''!\, H_k}{(H_k!)^2 \delta^2},$$

$$\frac{1}{6} \frac{\partial}{\partial v} \log \frac{\delta^4 H_{k-1}}{(\mathfrak{B}_{-(k-1)}!)^2 \mathfrak{B}_{-(k-1)} (H_{k-1}!)^2}$$

$$= \frac{1}{6} \frac{\partial}{\partial v} \log \frac{(H_k!)^4 \delta^4 \mathfrak{A}_{k+1}''}{(\mathfrak{A}_k''!)^2},$$

$$\frac{1}{6} \frac{\partial}{\partial u} \log \frac{(H_{k-1}!)^4 \delta^4 \mathfrak{B}_{-k}}{(\mathfrak{B}_{-(k-1)}!)^2} = \frac{1}{6} \frac{\partial}{\partial u} \log \frac{\delta^4 H_k}{(\mathfrak{A}_k''!)^2 \mathfrak{A}_k'' (H_k!)^2}, \tag{18}$$

$$H_{k-1} - \frac{1}{3} \frac{\partial^2}{\partial u \partial v} \log \frac{\mathfrak{B}_{-(k-1)}!\, H_{k-1}!}{\delta^2}$$

$$= H_k - \frac{1}{3} \frac{\partial^2}{\partial u \partial v} \log \frac{\mathfrak{A}_k''!\, H_k!}{\delta^2},$$

$$\frac{H_{k-1}}{\mathfrak{B}_{-(k-1)}} = \mathfrak{A}_{k+1}'',$$

$$- \frac{H_{k-1}}{\mathfrak{B}_{-(k-1)}} \cdot \frac{\partial}{\partial u} \log \frac{\mathfrak{B}_{-(k-2)}!\, H_{k-1}}{(H_{k-1}!)^2 \delta^2}$$

$$= - \mathfrak{A}_{k+1}'' \frac{\partial}{\partial u} \log \frac{\mathfrak{A}_{k+1}''!\, H_k!}{\delta^2}.$$

Since also $H_{-k} = H_{k+1}$, we have, in view of (5),

$$\frac{\partial^2}{\partial u \partial v} \log H_{k-1}!\, H_k! = 0, \tag{19}$$

so that by a suitable transformation of the independent variables we may make

$$H_{k-1}!\, H_k! = 1. \tag{20}$$

By use of (5) and (12), we find that substitution of (20) and the first of (18) in the remaining equations causes (18) to yield only one new condition, namely, $\mathfrak{A}_k''!\, H_{k-1}!/\mathfrak{B}_{-(k-1)}! =$ constant. We may make this constant equal to unity by a transformation of the independent variables, without violating the condition (20). Hence the following theorem results. *If an asymptotic plane net whose $(k+1)$th and minus $(k+1)$th*

transforms are not degenerate, is quasi-periodic, of period $2k+1$, *its differential equations may be so written that*

$$(21) \qquad \begin{gathered} H_{k-1}!\,H_k! = 1, \quad \mathfrak{A}_k''\mathfrak{B}_{-k} = H_k, \\ \mathfrak{A}_k''!\,H_{k-1}! = \mathfrak{B}_{-(k-1)}. \end{gathered}$$

Conversely, any asymptotic plane net, whose $(k+1)$*th and minus* $(k+1)$*th transforms are non-degenerate, and for which equations* (21) *hold, is quasi-periodic, of period* $2k+1$.

ANN ARBOR, MICH.,
April, 1916.

CONCERNING HILL'S DERIVATION OF THE LAGRANGE EQUATIONS OF MOTION.

BY PROFESSOR K. P. WILLIAMS.

(Read before the American Mathematical Society, April 22, 1916.)

THERE are two methods of deriving the Lagrange equations of motion that are commonly given in treatises on dynamics. One of the methods makes use of what is known as Hamilton's principle, while the other proceeds directly from D'Alembert's equation by means of a transformation of variables. While the first method leaves little to be desired as regards elegance, it makes use of a principle not essential to an understanding of the equations of motion or of their application. The second method, as usually given, involves a considerable amount of calculation.

In a paper entitled "On the differential equations of dynamics," in the first volume of the *Analyst*,* Hill sought to derive the Lagrange equations from D'Alembert's equation without making use of the details of the calculation above mentioned. For some reason his ideas do not seem to have found their way into the literature of the subject. In the form in which he presented it, Hill's derivation seems to me to be open to criticism on account of some of the assumptions that he makes. It is possible, however, to avoid making these assumptions, and when this is done a very simple and direct derivation of the Lagrange equations is obtained.

We start with D'Alembert's equation

* Collected Papers, vol. 1, pp. 192–194.

$$\sum_{i=1}^{n}\left[\left(m_i\frac{d^2x_i}{dt^2}-X_i\right)\delta x_i+\left(m_i\frac{d^2y_i}{dt^2}-Y_i\right)\delta y_i + \left(m_i\frac{d^2z_i}{dt^2}-Z_i\right)\delta z_i\right]=0,$$

where δx_i, δy_i, δz_i is any virtual displacement. Since

$$\frac{d}{dt}\delta u = \delta\frac{du}{dt} \qquad (u = x_i, y_i, z_i),$$

we obtain from this in the ordinary way

$$(1)\quad \frac{d}{dt}\sum_{i=1}^{n} m_i\left\{\frac{dx_i}{dt}\delta x_i+\frac{dy_i}{dt}\delta y_i+\frac{dz_i}{dt}\delta z_i\right\}-\delta(T-U)=0,$$

where T is the kinetic energy and $-U$ the force function. Let now $q_1, q_2, \cdots, q_k$ be the generalized coordinates of the system. Replacing x_i, y_i, z_i by their values in terms of the q's, the last equation takes the form

$$\frac{d}{dt}\sum_{i=1}^{k} p_i\delta q_i - \delta(\bar{T}-\bar{U})=0,$$

where $p_1, p_2, \cdots, p_k$ are quantities to be determined, and $\bar{T}$ and $\bar{U}$ denote what T and U, respectively, become after the substitution. At this point Hill says: "We can find the value of p_i without actually making the substitution from this consideration; since the original equation contains only the variations δx, δy, δz, etc., without the variations $\delta(dx/dt)$, $\delta(dy/dt)$, $\delta(dz/dt)$, it follows that, in the transformed state, it should contain only the variations δq_i without the variations $\delta(dq_i/dt)$." It scarcely seems legitimate to draw such a conclusion without further examination, especially since Hill expressly gives to the symbol $\delta(dx/dt)$, etc., no quantitative significance.* The hypothesis that must be made in order to evaluate p_i is obtained in the following way.

* The way in which Hill uses the symbol seems to me a little vague. Relative to the relation

$$\frac{d\delta x}{dt} = \delta\frac{dx}{dt} = \delta x'$$

he states: "The reader will see in this only a notational assumption, without quantitative significance, serving merely as machinery of demonstration." He expressly avoids using δx, δy, δz with the significance attached to them in the calculus of variations. This is legitimate in the case of D'Alembert's equation, where they may represent any virtual displacement, but they must be regarded as certain functions of t in equation (1).

The last equation when developed becomes

$$\sum_{i=1}^{k}\left(\frac{dp_i}{dt}\delta q_i + p_i\frac{d}{dt}\delta q_i\right) - \sum_{i=1}^{k}\left(\frac{\partial(\bar{T}-\bar{U})}{\partial q_i}\delta q_i + \frac{\partial \bar{T}}{\partial q_i'}\delta q_i'\right) = 0$$

or

$$(2)\quad \sum_{i=1}^{k}\left[\frac{dp_i}{dt} - \frac{\partial(\bar{T}-\bar{U})}{\partial q_i}\right]\delta q_i + \sum_{i=1}^{k}\left[p_i - \frac{\partial \bar{T}}{\partial q_i'}\right]\frac{d}{dt}\delta q_i = 0,$$

since

$$\delta q_i' = \frac{d}{dt}\delta q_i.$$

Suppose now that the system is holonomic. We can then give to the variations δq_i any values. Let us make each one of them constant throughout the motion, but not zero. The last summation in the equation above then vanishes, so that we have

$$\sum_{i=1}^{k}\left[\frac{dp_i}{dt} - \frac{\partial(\bar{T}-\bar{U})}{\partial q_i}\right]\delta q_i = 0.$$

Since $\delta q_1, \cdots, \delta q_k$ are all independent we then have at once

$$(3)\qquad \frac{dp_i}{dt} - \frac{\partial(\bar{T}-\bar{U})}{\partial q_i} = 0 \qquad (i = 1, 2, \cdots, k),$$

an equation entirely independent of the variations δq_i. Equation (2) then becomes

$$\sum_{i=1}^{k}\left[p_i - \frac{\partial \bar{T}}{\partial q_i'}\right]\frac{d}{dt}\delta q_i = 0.$$

Making all the quantities δq_i constant except in turn δq_1, $\delta q_2, \cdots, \delta q_k$, we have finally

$$p_i = \frac{\partial \bar{T}}{\partial q_i'}.$$

Upon substituting in (3) we have the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial q_i'}\right) - \frac{\partial(\bar{T}-\bar{U})}{\partial q_i} = 0 \quad (i = 1, 2, \cdots, k).$$

INDIANA UNIVERSITY.

A SIMPLIFICATION OF THE WHITEHEAD-HUNTINGTON SET OF POSTULATES FOR BOOLEAN ALGEBRAS.

BY DR. B. A. BERNSTEIN.

(Read before the San Francisco Section of the American Mathematical Society, November 20, 1915.)

Of the various sets of postulates that have been given for Boolean logic the most elegant and natural is the set of Huntington's based on Whitehead's "formal laws."* This set may be simplified by reducing the number of its postulates without injuring, the writer feels, the elegance or the naturalness of the original. This reduction is effected by substituting for Huntington's Postulates II_a, II_b, and V the following single postulate:

Postulate X. *For any element b in the class there exists an element $\bar{b}$ such that, whatever a is, $a \oplus (b \odot \bar{b}) = a$ and $a \odot (b \oplus \bar{b}) = a$.*

Evidently, Huntington's Postulates II_a, II_b, and V follow from Postulate X, with the help of I_a and I_b.

Evidently, also, Postulate X can be derived from II_a, II_b, and V, with the help of I_a, I_b, III_a, and III_b.

It is of course seen that by adopting Postulate X in place of II_a, II_b, and V, not only is the number of Huntington's postulates reduced from ten to eight, but also the number of postulated special elements is reduced from three ("zero," the "whole," and the "negative") to one (the "negative").

In establishing the independence of the modified set of postulates Huntington's systems for I_a, I_b, IV_a, IV_b, VI can serve for the same numbered postulates in the new set. For Postulate X we can take Huntington's system for V. For III_a and III_b, however, a class of more than two elements is, in each case, necessary. Proof-systems for these two postulates are, respectively, the following:

$\overline{\mathrm{III}_a}$.

$\oplus$	e_1	e_2	e_3
e_1	e_1	e_1	e_1
e_2	e_1	e_2	e_2
e_3	e_1	e_3	e_3

$\odot$	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	e_2	e_2
e_3	e_3	e_2	e_3

* See E. V. Huntington, "Sets of independent postulates for the algebra of logic," *Transactions Amer. Math. Society*, vol. 5 (1904), pp. 288–309. The set referred to is the first of the three sets treated by Huntington in his paper.

Here $e_2 \oplus e_3 \neq e_3 \oplus e_2$.

$\overline{\text{III}}_b$.

$\oplus$	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	e_2	e_2
e_3	e_3	e_2	e_3

$\odot$	e_1	e_2	e_3
e_1	e_1	e_1	e_1
e_2	e_1	e_2	e_2
e_3	e_1	e_3	e_3

Here $e_2 \odot e_3 \neq e_3 \odot e_2$.

UNIVERSITY OF CALIFORNIA,
March, 1916.

NOTE ON REGULAR TRANSFORMATIONS.

BY DR. L. L. SILVERMAN.

LET $u(x)$ be bounded and integrable, $0 \leqq x$, and $k(x, y)$ integrable in y for each x, $0 < y \leqq x$; then the transformation*

$$(1) \qquad v(x) = \alpha u(x) + \int_0^x k(x, s)u(s)ds$$

is regular if

$$\lim_{x=\infty} u(x)$$

implies the existence of

$$\lim_{x=\infty} v(x)$$

and the equality of the limits. The transformation (1), which depends on the number α and on the function $k(x, y)$, will be denoted by the symbol $[\alpha; k(x, y)]$. Examples of regular transformations are given by $[1; 0]$, which is the identical transformation, and $[0; 1/x]$, which corresponds to the first Hölder mean. In a forthcoming paper† the author discusses conditions on α and $k(x, y)$ for the regularity of the transformation‡ (1), and proves the following theorem:||

THEOREM 1. *A sufficient condition that $k(x, y)$ defined, $0 < y \leqq x$, and integrable in y for each x, correspond to a*

---

* It is assumed that the improper integral converges; the lower limit of integration is taken zero for convenience.

† *Transactions*, vol. 17 (1916).

‡ The function $k(x, y)$ in (1) is $(1 - \alpha)$ times the function $k(x, y)$ in the article referred to.

|| See Theorem III in the article referred to; the numbers a and b of that theorem are here replaced by 0 and a respectively. The right-hand member of the last condition is $1 - \alpha$ instead of unity; see preceding footnote.

regular transformation is

$$\int_0^a |k(x, y)|dy \textit{ converges,} \quad \lim_{x=\infty} \int_0^a |k(x, y)|dy = 0,$$

$$\int_0^x |k(x, y)|dy < A, \quad x > 0, \quad \lim_{x=\infty} \int_0^x k(x, y)dy = 1 - \alpha,$$

where a and A are positive constants.

We wish in this note to consider some special cases of Theorem 1.

THEOREM 2. *Let $k(x, y)$ be defined, $0 < y \leqq x$, and integrable in y for each x; then a sufficient condition* that the transformation $[\alpha; k(x, y)]$ be regular is that*

$$|k(x, y)| \leqq \frac{Mf'(y)}{f(x)}, \quad 0 < y \leqq x, \quad \lim_{x=\infty} \int_0^x k(x, y)dy = 1 - \alpha,$$

where $f(x)$ is a function continuous, $x \geqq 0$, and having a continuous derivative, $x > 0$, and satisfying the conditions

$$f(x) \geqq 0, \quad x \geqq 0; \quad f'(x) \geqq 0, \quad x > 0; \quad \lim_{x=\infty} f(x) = \infty;$$

and where

$$M \geqq 0.$$

The fourth condition of Theorem 1 is satisfied by hypothesis. The first condition is satisfied since from the hypothesis it follows that the integral of $k(x, y)$ converges absolutely. Furthermore,

$$\int_0^a |k(x, y)|dy \leqq \frac{Mf(a)}{f(x)}, \quad x > 0.$$

The second and third conditions of Theorem 1 follow at once from this inequality. Thus all the conditions of Theorem 1 are satisfied.

COROLLARY 1. *A sufficient condition that $k(x, y)$ defined, $0 < y \leqq x$, and integrable in y for each x, correspond to a regular transformation is*

$$|k(x, y)| \leqq \frac{M}{(1 + y) \log (1 + x)}, \quad 0 < y \leqq x,$$

$$\lim_{x=\infty} \int_0^x k(x, y)dy = 1 - \alpha,$$

where $M \geqq 0$.

* The convergence of the integral in the second condition—in fact, its absolute convergence—follows from the first condition.

Taking $f(x) = \log(1 + x)$, it is seen that the conditions of Theorem 2 are satisfied.

COROLLARY 2. *A sufficient condition that $k(x, y)$ defined, $0 < y \leqq x$, and integrable in y for each x, correspond to a regular transformation is*

$$|k(x, y)| \leqq \frac{M}{x^{1-p}y^p}, \quad 0 < y \leqq x, \quad \lim_{x=\infty} \int_0^x k(x, y)dy = 1 - \alpha,$$

where p and M are constants, $0 \leqq p < 1$, $M \geqq 0$.

Taking $f(x) = x^{1-p}$, it is seen that the conditions of Theorem 2 are satisfied.

Similar theorems hold for transformations of sequences.

CORNELL UNIVERSITY.

SHORTER NOTICES.

Contributions to the Founding of the Theory of Transfinite Numbers. By GEORG CANTOR. Translated, and provided with an introduction and notes, by PHILIP E. B. JOURDAIN. Chicago and London, Open Court Publishing Company, 1915. ix+211 pp.

THIS volume contains a translation of G. Cantor's two fundamental memoirs on transfinite numbers which appeared in the *Mathematische Annalen* for 1895 and 1897 under the title "Beiträge zur Begründung der transfiniten Mengenlehre." The translator has changed the title to that given above because "these memoirs are chiefly occupied with the investigation of the various transfinite cardinal and ordinal numbers." The book is put forth by the publishers as number 1 of "The Open Court Series of Classics of Science and Philosophy."

It is not too much to say that the work of Cantor on the theory of classes of points has brought about both a mathematical and a philosophical revolution, that in philosophy perhaps being even greater than that in mathematics, notwithstanding the fact that "these theories of Cantor are permeating modern mathematics."*

In the opinion of the translator K. Weierstrass, R. Dedekind, and G. Cantor are the three men who have exerted the most marked influence on modern pure mathematics and indirectly on the modern logic and philosophy which abut on it. In

* E. H. Moore, Introduction to a Form of General Analysis, p. 2.

order that the work of Cantor—and, in particular, the two memoirs here translated—may be best understood it is desirable on the one hand to compare it with the work of Dedekind which has developed along a parallel direction and on the other hand to trace its origin backwards through the earlier theory of functions and especially through the work of Weierstrass. To facilitate the latter historical study this book is provided with an excellent introduction (pages 1–82), based in large part on the translator's "Development of the theory of transfinite numbers," published in the *Archiv der Mathematik und Physik* in 1906, 1909, 1910, and 1913.

This introduction begins with a brief account of the contribution of Fourier toward the general definition of function, followed by remarks concerning the labors of Dirichlet, Cauchy, Riemann, and Hankel in the theory of functions as well as by some further discussion of the theory of Fourier series. A fuller account is then given (pages 10–23) of the contributions of Weierstrass to the theory of functions. The remainder of the introduction (pages 23–82) is devoted to an illuminating account of the development of Cantor's ideas as seen from his publications prior to 1895.

The translation of Cantor's articles covers pages 85–201. On the whole the work of the translator is done in a satisfactory manner. One regrets, however, that the term "power" is used to translate both "Potenz" and "Mächtigkeit," thus introducing an ambiguity in the English which is unnecessary and is absent from the German. Sometimes (as on pages 97, 178) the translator finds it necessary to introduce a footnote to remove the ambiguity.

Brief notes at the end (pages 202–208) contain a short but valuable account of the development of the theory of transfinite numbers since 1897.

A few misprints and slips need correction, as follows: on page 16, line 5 below for s_{n+} read $s_{n+\gamma}$; on page 31, lines 8 and 9, for P', P'', P read P'', P''', P'; on page 33, line 8 below, for *consists of* read *contains*; on page 47, end of line 13, insert *at*; on page 69, line 1, for *in* read *is*; at the middle of page 71, for last exponent 3 read 2; on page 96, line 4, after $x \geqq$ insert 0; on page 96, line 9, for $2^{\aleph}$ read $2^{\aleph_0}$; on page 109, line 12 below, for $\aleph$ read $\aleph_0$; on page 184, line 3 below, for the first ω read ω^{α_0}; on page 194, line 3, for $\alpha^{(\rho_0}$ read $\alpha^{(\rho_0)}$; on page 196, line 11 below, for ω^{γ} read ω^{γ_ν}.

In making this fundamental work of Cantor readily accessible to a wider range of English readers both the translator and the publishers have rendered a useful service in the development of science.

R. D. CARMICHAEL.

Euclid's Book on Divisions of Figures (περὶ διαιρέσεων βιβλίον), with a restoration based on Woepcke's text and on the Practica Geometriæ of Leonardo Pisano. By RAYMOND CLARE ARCHIBALD. Cambridge, Eng., University Press, 1915. ix+88 pp.

OF the nine works attributed to Euclid the "Elements" is, of course, by far the most important and the most widely known. The "Data" is known to us through the τόπος ἀναλυόμενος of Pappus, as stated in the Commandino edition of 1660, page 241; the "Porisms" was restored by Chasles, and earlier by Robert Simson; the "Optics" was known to earlier scholars through Theon, and has recently appeared in a modern edition through the labors of Heiberg; the "Phænomena" is nearly complete and was edited by Menge; the "Conics" is lost, except as part of it may have been embodied in the works of Apollonius; and the "Pseudaria" and "Surface Loci" are known only through fragments. The ninth work, entitled "On Divisions" (of figures), was for a long time known only through references by Proclus, but in 1570 it appeared in print under the editorship of John Dee and Federico (sometimes printed Federigo) Commandino in Latin translation from the Arabic. In 1851 Woepcke found an Arabic manuscript of the work at Paris, and this was published in translation in the *Journal Asiatique*.

It seems that John Dee, when he visited Commandino at Urbino in 1563, gave to the latter a Latin manuscript of the work as translated into Arabic by one Muhammed Bagdedinus, and this together with an Italian version was published seven years later. An English translation appeared in London in 1660 and again in 1661. David Gregory included the Latin text in his edition of Euclid in 1703 with the statement: "Joannes Dee Londinensis, cum Librum de Divisionibus superficierum, Machometo Bagdedino (qui floruisse creditur seculo Christi decimo) vulgo adscriptum, ex Arabico (uti credo, licet hoc expresse non dicat) in Latinum verteret." As to the conjectured date of "Machometo Bagdedino" it may be said in

passing, that Professor Archibald believes, with Suter, that he was the Muhammed of Bagdad who died in 1141, and hence that he was a scholar of the eleventh and twelfth centuries.

Professor Archibald has, with great perseverance and scholarship, cleared up a number of points in connection with this translation. In the first place he has shown that Suter and Steinschneider have not considered their statements concerning it with their usual care. For example, it has been commonly asserted that Dee probably copied a Latin translation in the Cottonian collection, whereas it is here shown that he did not do so, and that no such translation in complete form was ever in the British Museum. It is very doubtful whether a translation made by Gherardo of Cremona was the one referred to in the Cottonian catalogue made by Smith in 1696. At any rate this particular manuscript was not in Gherardo's handwriting since it was of about the fourteenth century. The fact is that Dee owned a manuscript of the work itself, possibly a copy of Gherardo's translations, and very likely he owned another besides the one which (maybe as a duplicate) he gave to Commandino.

Professor Archibald's plan in editing the work was to translate literally everything in Woepcke's French translation from the Arabic manuscript in the Bibliothèque nationale, to reproduce Fibonacci's proofs and constructions as set forth in his "Practica Geometriæ" of 1220, and to show the correspondence of the Muhammed-Commandino treatise with the Euclid text and with Fibonacci. He has also shown the relation of the "Geometria vel De Triangulis" of Jordanus Nemorarius to the work in question, revealing some interesting and significant facts. It would be well worth the attention of some scholar to consider the "De Numeris Datis" in the same spirit.

Of the work itself this is not the place to speak, further than to say that it has to do with such divisions of plane figures as the separation of a given triangle into two equal parts by a line which passes through a point situated in the interior of the triangle, and to call attention to the fact that this is the source of a number of such problems, some of which played a considerable rôle in the older treatises on surveying. But of the work of editing the text it may be said that it is a perfectly appropriate compliment to pay both to Professor Archibald and to Sir

Thomas Heath to say that the care shown by each, in the editing of the classics to which their names attach, is on a par with that shown by the other. Certainly we have not had any such work done before in this country in the editing of any Greek mathematical text, and the thanks and appreciation of Professor Archibald's colleagues will go out to him in abundant measure for his excellent contribution to the literature of the subject.

Not the least of the commendable features of the edition under review is the bibliography of works relating to the division of figures. How complete this is it is difficult to say, since one would have to go through a large amount of material, as has evidently been done in this case, to determine just where to find points of contact with the "De Divisionibus." At any rate the list is a very helpful one and adds materially to the value of the work.

The publishers have allowed the printing of a copious index of names, and have issued the work in the dignified form which always characterizes the output of the Cambridge University Press.

DAVID EUGENE SMITH.

Analytic Geometry. By H. B. PHILLIPS. New York, John Wiley and Sons, 1915. vii+197 pp.

IN the preface of this book the author expresses the belief that for engineering students "a short course in analytic geometry is essential"; and "he has, therefore, written this text to supply a course that will equip the student for work in calculus and engineering without burdening him with a mass of detail useful only to the student of mathematics for its own sake." At first glance the comparatively small number of pages would seem to promise such a short course. But a closer examination led the present writer to the opinion that the apparent brevity was achieved by condensation, and that it would require as much time to cover these 197 pages as to cover say 300 pages of many other texts. Except for the omission of some of the special properties of the conics, it did not seem quite obvious that the student's burden of a mass of detail was conspicuously lightened.

On the other hand, the author at times assumes a clearness of mathematical vision and a facility in technique on the part of the student which would be eminently desirable, but

which we fail to find in a majority of our college students. For instance, after learning (page 94) that "a curve is symmetrical with respect to the origin if all the terms in its equation are of even degree or if all are of odd degree," we find the following example (I quote the text exactly):

"$y = x^3 - 3x^2 + 3x + 1$. This equation can be written

$$y - 2 = (x - 1)^3.$$

The expressions $y - 2$ and $x - 1$ are the coordinates of a point $P(x, y)$ relative to the lines $y = 2$, $x = 1$ used as axes. Since the equation contains only odd powers of $y - 2$ and $x - 1$, the curve is symmetrical with respect to the point (1, 2)." This occurs 50 pages earlier than the treatment of transformations to parallel axes. At such points the ordinary student will certainly need the help of the teacher. However, many teachers will find this no objection to the book, but will prefer to use a text which leaves to the teacher some further function than the assigning of lessons and the conducting of recitations. Teachers with ideas of their own are sometimes hampered by a text too liberal in its explanations.

The first chapter, on algebraic principles, furnishes a good review of those parts of algebra which are most necessary for what follows. One is pleased to find a discussion of inequalities and of homogeneous linear equations. In the second chapter, introducing rectangular coordinates, there are six excellent pages on vectors. The straight line and circle are treated in chapter three, which is short but would seem to furnish the student with the necessary working equipment. A very good feature is the clear exposition of the geometrical significance of the sign of the expression $Ax + By + C$; but it is to be regretted that the idea was not used to the best advantage later; as, for instance, in connection with the distance formula $(Ax_1 + By_1 + C)/ \pm \sqrt{A^2 + B^2}$, and the definition of the hyperbola.

After a somewhat unusual treatment of the conics in chapter four, one finds a very interesting chapter on graphs and empirical equations. In addition to the usual curve drawing, the problem of fitting a curve to a number of given pairs of values of the variables is discussed, with a number of exercises drawn from mechanics, electricity, chemistry, etc. Chapters six, seven, and eight are devoted respectively to polar coordinates, parametric representation, and transformations. The

chapter on parametric representation, with the development of the equations of the cycloidal and other curves, is one of the excellent features of the book. The last three chapters present very briefly the first elements of the analytic geometry of space, treating the plane, straight line, sphere, cylindrical surfaces, surfaces of revolution, and the different quadric surfaces in their simplest positions. The helix is discussed in a very short paragraph on parametric equations of space curves.

The author's treatment of conics should have special mention. The ellipse, parabola, and hyperbola are defined as follows:

"If a circle is deformed in such a way that the distances of its points from a fixed diameter are all changed in the same ratio, the resulting curve is called an ellipse.

"Let LK, RS be perpendicular lines and MP, NP perpendiculars from any point P to them. If a is constant and NP considered positive when P is on one side of RS, negative when on the other, the locus of points P such that

$$MP^2 = a \cdot NP$$

is called a parabola.

"Let KL and RS be two straight lines intersecting in C, PM and PN the perpendiculars from any point P to these lines. Let MP be considered positive when P is on one side of KL, negative when on the other side. Similarly, let NP be positive when P is on one side of RS, negative when on the other side. A hyperbola is the locus of points P such that the product

$$MP \cdot NP = \text{constant.}"$$

The general definition of a conic, and the classification with respect to the eccentricity are given later in the chapter on polar coordinates. There is no question here of correctness or incorrectness, but merely one of taste and expediency. Many teachers would prefer that the conics should be first introduced by a general definition; and many others, willing to use independent definitions of the three types, would prefer other definitions than those which the author has seen fit to use. Immediately following the definition of the parabola we read: "A parabola is thus a locus of points the squares of whose distances from one of two perpendicular lines are proportional to their distances from the other. The complete locus of

such points is two parabolas, one on each side of RS." A similar statement follows the definition of the hyperbola. Is it not unfortunate to speak of a locus which is not a complete locus?

The definition of equivalence of sets of equations (page 3) is somewhat vague; and it hardly seems wise to say that the equation $(x+y)(x-2y)=0$ is equivalent to the two equations $x+y=0$ and $x-2y=0$, even though the sense in which this is meant is immediately explained.

The statement (page 15) that "a set of homogeneous equations can often be solved for the ratios of the variables when there are not enough equations to determine the exact values" might seem to imply that the "exact" values could be determined if there were enough equations.

In chapter five there is a paragraph on "infinite values" which reminds one of the school algebras of the last generation. It seemed to the present writer to be a really serious defect in what is in many respects an excellent book.

The mechanical features of the book are attractive, the figures (with a few exceptions) are accurate, and the typographical work is free from errors.

WALTER B. CARVER.

A Budget of Paradoxes. By AUGUSTUS DE MORGAN. Reprinted, with the author's additions, from the *Athenæum.* Second edition, edited by DAVID EUGENE SMITH. Two volumes, I, viii+402 pp.; II, 387 pp. Chicago, The Open Court Publishing Co., 1915. Price, $3.50 per volume.

THE first edition of this interesting work by Augustus De Morgan (1806–1871) appeared in 1872, after the author's death, under the editorship of his widow, Sophia De Morgan. Some ten years later Mrs. De Morgan wrote a "Memoir of Augustus De Morgan," which is worthy of mention in connection with the "Budget of Paradoxes." De Morgan's articles which constitute the present work appeared from time to time, in the years from 1863 (Oct. 10) to 1866 (Dec. 1), in the London *Athenæum.* From other facts which we have concerning the life of De Morgan it appears that some of the popular writing which he did, for encyclopedias and for journals, was stimulated by financial pressure; at this distance we can properly rejoice at the conditions which fostered the growth of the present work.

The wide range of interests of De Morgan is nowhere so well shown as in the somewhat random discussions of the "Budget of Paradoxes." In particular his interest in the history of mathematics, and in the history of science, is revealed again and again in these pages. De Morgan holds that a reasonable familiarity with the development of any field of scientific research is an indispensable, necessary condition for a contribution to the field. "All the men who are now called discoverers, in every matter ruled by thought, have been men versed in the minds of their predecessors, and learned in what had been before them. There is not one exception. I do not say that every man has made direct acquaintance with the whole of his mental ancestry; many have, as I may say, only known their grandfathers by the report of their fathers. But even on this point it is remarkable how many of the greatest names in all departments of knowledge have been real antiquaries in their several subjects.

"I may cite, among those who have wrought strongly upon opinion or practise in science, Aristotle, Plato, Ptolemy, Euclid, Archimedes, Roger Bacon, Copernicus, Francis Bacon, Ramus, Tycho Brahé, Galileo, Napier, Descartes, Leibnitz, Newton, Locke. I take none but names known out of their fields of work; and all were learned as well as sagacious. I have chosen my instances: if any one will undertake to show a person of little or no knowledge who has established himself in a great matter of pure thought, let him bring forward his man, and we shall see."

Taking into account the special interests of De Morgan and his great activity in the popularizing of mathematics the selection of David Eugene Smith as editor of this edition was almost inevitable. Between the time of the English mathematician and the present time no one could have been found more admirably fitted by nature and by training to edit the "Budget" than Professor Smith. The similarity of his literary and public activity to De Morgan's is striking; both men have been widely known as the authors of elementary textbooks of unusual excellence, both have acted as editors of the mathematical department of encyclopedias and dictionaries, both have been energetic collectors of old mathematical books and other mathematical material, both have made notable contributions to the history and bibliography of mathematics, and both have been distinguished by a wide and human

interest in mathematics. Smith's "Rara Arithmetica," the illuminating, descriptive catalogue of Mr. G. A. Plimpton's unparalleled collection of arithmetical books and manuscripts, is a continuation and extension of De Morgan's bibliographical work, "Arithmetical Works from the Invention of Printing to the Present Time" (London, 1847).

The intention of De Morgan in publishing the "Budget" was definitely stated to be "to enable those who have been puzzled by one or two discoverers to see how they look in a lump." By "discoverers," here, is meant paradoxical discoverers. For De Morgan the paradox "is something which is apart from general opinion, either in subject-matter, method, or conclusion." That no disparagement is implied necessarily in the designation "paradoxer" is indicated by the fact that De Morgan refers to Copernicus and Galileo as paradoxers, and includes the discovery of the planet Neptune by Le Verrier as a paradox. A distinction is drawn between two types of paradoxer, as follows: "The manner in which a paradoxer will show himself, as to sense or nonsense, will not depend upon what he maintains, but upon whether he has or has not made a sufficient knowledge of what has been done by others, *especially as to the mode of doing it*, a preliminary to inventing knowledge for himself." However it must be stated that the particular interest of De Morgan in this work is in the nonsense type of paradox and paradoxer.

In mathematics the old problems of the squaring of the circle, the duplication of the cube, the trisection of the angle, and numerical juggling with 666, the number of the beast, contribute most largely to the occupation of neophytes who "in a moment by a lucky thought" wish to enter their names upon the limited roll of great mathematicians. Others, and their kind is not yet extinct, have the notion that either at home or abroad there is a great reward offered for the squaring of the circle; some even have deluded themselves into thinking that this financial reward is involved in a university professorship. What could be more illuminating as to the ignorance of these paradoxers! Astronomy contributes its fair quota to the "Budget"; religion, philosophy, and medicine, too, have their pseudo-scientists who without knowing "what has been done by others" wish to revolutionize the established order. Would that all this were, indeed, a closed book. In mathematics trisectors are fairly common, and have persuaded journals,

even with scientific titles, to publish nonsense; on Fermat's theorem several hundred articles have been printed and others are now being written by paradoxers whose tense interest in the subject is occasioned rather by a certain familiarity with the power of the $25,000, an actual prize, than by any familiarity with the powers of integers; what nonsense emanates even from a distinguished seat of learning—absolutely, let us note, without official sanction—concerning a prodigy lecturing upon the fourth dimension; what a priori philosophical nonsense, based upon ignorance of "what has been done by others," has been published concerning the nature of the number idea. A modern De Morgan would, in two volumes like these, have room only for titles of published nonsense.

The present edition of the "Budget" will prove of considerable value to public libraries as a work of reference; the value would be greatly increased by an analytical table of contents, presenting the titles of the articles comprising the work, and giving, possibly, a summary of the articles arranged with reference to subject-matter. Mathematicians and astronomers have a particular interest in the work since so much of the material is germane to their fields. To that wide circle of readers who have enjoyed the artistic ramblings of William De Morgan this work by the famous novelist's father will prove entertaining, for the peculiar literary charm of the son seems to be a direct transmission from the father.

Typographically and otherwise the book is up to the high standard which has been set by the publications of the Open Court Company. The reader who takes the volumes in hand has real pleasure in store: we commend the work to all of those who take a kindly interest in the frailty, as well as in the greatness, of their fellows.

The errors of one sort and another are so difficult to find that it seems desirable to mention those which have caught the eye of the reviewer. In the Preface, I, page iv, the reference in the last line should be to page 280, not 281, of the second volume. The one good "i" is dropped out of an "acquaintance" on page 5, volume I. From Smith's "Rara Arithmetica" three or four errors should be corrected in the title of Robert Recorde's "The Whetstone of Witte" (II, page 328): The whetstone of witte . . . containyng . . . Cossike . . . Nombers. The date of death of Sacrobosco, given as 1256, (I., page 360) is not known. LOUIS C. KARPINSKI.

NOTES.

The March number (series 2, volume 17, number 3) of the *Annals of Mathematics* contains the following papers: "An isomorphism between theta characteristics and the $(2p+2)$-point," by A. B. Coble; "On certain real solutions of Babbage's functional equation," by J. F. Ritt; "Note on the preceding paper," by A. A. Bennett; "An elementary exposition of the theory of the gamma function" (authorized translation from the Danish by T. H. Gronwall), by J. L. W. V. Jensen.

The first four numbers of volume two of the *Proceedings of the National Academy of Sciences* contain: "Upper limit of the degree of transitivity of a substitution group," by G. A. Miller; "An extension of Feuerbach's theorem," by Frank Morley; "Deformations of transformations of Ribaucour," by L. P. Eisenhart; "On the linear dependence of functions of several variables, and certain completely integrable systems of partial differential equations," by G. M. Green; "Point sets and allied Cremona groups (part II)," by A. B. Coble; "On a theorem of Lucas," by M. B. Porter; "Interpretation of the simplest integral invariant of projective geometry," by E. J. Wilczynski.

University of Chicago.—The following courses are announced for the summer quarter, June 19–September 1: By Professor G. A. Bliss: Theory of functions of a real variable.—By Professor L. E. Dickson: Substitution groups and algebraic equations, solution of numerical equations (first half); Determinants and symmetric functions (second half).—By Dr. C. R. Dines (of Dartmouth College): Differential equations.—By Professor W. D. MacMillan: Introduction to celestial mechanics.—By Professor E. H. Moore: Integral equations in general analysis (first half); Limits (second half).—By Professor F. R. Moulton: Theory of functions of infinitely many variables (second half); Series (second half).—By Professor A. Ranum (of Cornell University): Metric differential geometry.—By Professor H. E. Slaught: Elliptic integrals.—By Professor J. W. A. Young: Selected topics in mathematics.

THE following advanced mathematical courses are announced for the academic year 1916–1917:

UNIVERSITY OF CHICAGO.—By Professor G. A. BLISS: Theory of functions of a real variable (a); Calculus of variations (w); Functions of lines (s).—By Professor L. E. DICKSON: Theory of numbers (a); Algebraic numbers (w); Linear algebra (s); Theory of equations (w); Solid analytics (s).—By Professor K. LAVES: Analytic mechanics (a, w).—By Professor A. C. LUNN: Vector analysis (a); Applications to electromagnetics (w); Advanced calculus (a); Geometric introduction to the theory of the complex variable (w).—By Professor W. D. MACMILLAN: Introduction to celestial mechanics (w, s).—By Professor E. H. MOORE: Integral equations in general analysis (a, w, s); Seminar on the foundations of analysis (a, w).—By Professor F. R. MOULTON: Modern theories of analytic differential equations with applications to celestial mechanics and periodic orbits (a, w); Lunar theory (w); Application of the method of periodic orbits to the lunar theory (a).—By Professor H. E. SLAUGHT: Differential equations (a).—By Professor E. J. WILCZYNSKI: Projective geometry (a, w).—By Professor J. W. A. YOUNG: Limits and series (s).

COLUMBIA UNIVERSITY.—By Professor T. S. FISKE: Theory of functions, four hours.—By Professor M. W. HASKELL: Higher plane curves, three hours, first semester; Continuous groups, three hours, first semester.—By Professor F. N. COLE: Algebra, four hours.—By Professor JAMES MACLAY: Theory of numbers, three hours.—By Professor D. E. SMITH: History of mathematics, three hours; Seminar in the teaching of mathematics.—By Professor EDWARD KASNER: Contact transformations, two hours, second semester; Seminar in differential geometry.—By Professor W. B. FITE: Divergent series, three hours, second semester.—By Professor H. E. HAWKES: Differential geometry of curves, three hours, second semester.

JOHNS HOPKINS UNIVERSITY.—By Professor F. MORLEY: Higher geometry (first term), three hours; Theory of functions (second term), three hours.—By Professor A. B. COBLE: Finite groups, two hours; Probabilities (second term), two hours.—By Professor A. COHEN: Theory of numbers, two hours; Theory of functions, two hours.—By Dr. H. BATEMAN: Differential equations of physics, two hours.

University of Illinois.—By Professor E. J. Townsend: Functions of real variables, three hours; Differential equations, three hours.—By Professor G. A. Miller: Elementary theory of groups, three hours; Theory of equations, three hours, first semester.—By Professor H. L. Rietz: Actuarial theory, three hours.—By Professor J. B. Shaw: Vector methods, three hours.—By Professor C. H. Sisam: Invariants and higher plane curves, three hours; Solid analytic geometry, three hours, second semester.—By Professor A. Emch: Automorphic functions, three hours.—By Professor R. D. Carmichael: Theory of linear differential equations, three hours. —By Professor A. R. Crathorne: Projective geometry, three hours.—By Dr. E. B. Lytle: History of mathematics, two hours, first semester.—By Dr. A. J. Kempner: Modern algebra, three hours.

Dr. George Sarton, of Ghent, editor of *Isis*, has been appointed lecturer on the history of science at Harvard University. In the academic year 1916–1917 he will give two courses, one on "The origin and development of Greek science" and the other on the "Principles of mathematics historically considered." In the year 1917–1918 he will lecture on Hellenic science and on the principles of mechanics historically considered. He will also lecture on the history of science at the Lowell Institute in Boston.

The Smith prizemen for the year 1916, announced by the University of Cambridge, are as follows: H. M. Garner, of St. John's College, for his essay, "On orbital oscillation about the equatorial triangular configuration in the problem of three bodies"; G. P. Thompson, of Corpus Christi College, for his essay, "On aeroplane problems."

The Rayleigh prize for 1916 has been awarded to W. M. Smart, of Trinity College, for his essay, "Libration on the Trojan planets."

Professor C. J. de la Vallée-Poussin, of the University of Louvain, has been elected to membership in the Paris academy of sciences.

Professor G. A. Bliss, of the University of Chicago, has been elected a member of the National academy of sciences.

Professors M. Bôcher, of Harvard University, and F. R. Moulton, of the University of Chicago, have been elected to membership in the American philosophical society. Professor J. D. v. d. Waals, of Amsterdam, has been elected a foreign member of the society.

Professor C. J. Keyser, of Columbia University, and Professor M. W. Haskell, of the University of California, will exchange professorships during the first term of 1916–1917.

Mr. R. W. Dickey has been appointed associate professor of mathematics at Washington and Lee University.

Dr. J. I. Tracy, of Yale University, has been promoted to an assistant professorship of mathematics.

Dr. F. J. McMackin, of Columbia University, has been appointed instructor in mathematics at Dartmouth College.

At the University of Oklahoma Mr. C. T. Levy, of the University of California, has been appointed instructor in mathematics. Mr. E. D. Meacham, instructor in mathematics, will spend a year's leave of absence in study at Harvard University.

Professor Webster Wells, for many years instructor and professor of mathematics in the Massachusetts Institute of Technology and author of several mathematical textbooks, died at Boston on May 23 at the age of sixty-five years. Professor Wells retired from active service in 1911. He was a member of the American Mathematical Society from 1891.

Catalogues of books:—Carnegie Institution of Washington, List of publications to March 1, 1916.—W. Heffer and Sons, Cambridge, England, catalogue 148, including selections from the library of the late E. J. Routh.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BECKER (O.). Ueber die Zerlegung eines Polygons in exklusive Dreiecke auf Grund der ebenen Axiome der Verknüpfung und Anordnung. Leipzig, 1914. 8vo. 71 pp. M. 2.00

BIEBERBACH (L.). Einführung in die konforme Abbildung. Leipzig, Göschen, 1915. M. 0.90

——. Ueber die Grundlagen der modernen Mathematik. Leipzig, 1914. 4to. 6 pp.

DARBOUX (G.). Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal. Deuxième édition, 2e partie. Paris, Gauthier-Villars, 1915. Royal 8vo. 8+580 pp. Fr. 20.00

DINI (U.). Calcolo integrale. 1a e 2a parte. Pisa, Nistri, 1909–1915. 13+1060 pp.

DUNKEL (O.). See GOURSAT (E.).

FRICKE (R.). Die elliptischen Funktionen und ihre Anwendungen. In 3 Teilen. 1ter Teil: Die funktionentheoretischen und analytischen Grundlagen. Leipzig, Teubner, 1916. 8vo. 10+500 pp. Leinwand. M. 24.00

GOURSAT (E.). A course in mathematical analysis. Functions of a complex variable, being Part 1 of Volume 2. Translated from the second French edition by E. R. Hedrick and O. Dunkel. Boston, Ginn, 1916. 8vo. 10+259 pp. $2.75

HAMBURGER (H.). Ueber die Integration linearer homogener Differentialgleichungen. München, 1914. 4to. 69 pp.

HARST (J. H. VAN DER). Eenige ontaardingen van den harmonischen complex. Leiden, 1915. 8vo. 137 pp.

HEDRICK (E. R.). See GOURSAT (E.).

HOORN (J. VAN). Over de transfinite getallen en de leer van het kontinuum. Leiden, 1915. 4to. 111 pp.

HURWITZ (L.). Ueber den Zusammenhang der Drehungen des Raumes mit dem Kugelkreise. Königsberg, 1914. 8vo. 77 pp.

KILLERMAN (A.). Beweis des Fermatschen Satzes. Nürnberg, 1916. M. 3.00

KLEIN (F.). Bericht über den heutigen Zustand des mathematischen Unterrichts an der Universität Göttingen. Leipzig, Teubner, 1915. M. 0.40

KNOTT (C. G.). See NAPIER.

LÉRY (G.). Sur la fonction de Green pour un contour algébrique. Paris, Gauthier-Villars, 1915. 4to. 88 pp. Fr. 5.00

NAPIER tercentenary memorial volume; being papers contributed to the tercentary congress held at Edinburgh in July 1914. Edited by C. G. Knott. New York, Longmans, 1915. 4to. 12+441 pp. $7.00

PFLÜGER (J.). Die Formschönheit einfacher geometrischer Gebilde. Stuttgart, Metzler, 1915.

PICARD (E.). L'histoire des sciences et les prétensions de la science allemande. (Pour la Vérité.) Paris, Perrin, 1916. 16mo. 49 pp.

PINCHERLE (S.). Algebra complementare. Parte II: Teoria delle equazioni. 3a edizione, riveduta e corretta. (Manuali Hoepli, serie scientifica, No. 145.) Milano, Hoepli, 1916. 24mo. 167 pp. L. 1.50

——. Lezioni di calcolo infinitesimale. Bologna, Zanicelli, 1915. 15+774 pp.

RAPP (P.). Ueber Gruppen vom Rang Null. Tübingen, 1914. 8vo. 35 pp.

ROTTSIEPER (W.). Graphische Lösung einer Randwertaufgabe der Gleichung $\Delta u = \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$. Göttingen, 1914. 8vo. 45 pp. M. 2.00

SCHMIDT (C. P.). Kulturhistorische Beiträge zur Kenntnis des griechischen und römischen Altertums. 1tes Heft: Zur Entstehung und Terminlogie der elementaren Mathematik. 2te verbesserte und stark vermehrte Auflage. Leipzig, Dürrsche Buchhandlung, 1914. 8vo. 16+269 pp.

TORHORST (A.). Die rationale Normalfläche vierter Ordnung des R_5 und ihre Projektionen in den vier- und dreidimensionalen Raum. (Diss.) Bonn, 1915.

VOOREN (W. L. VAN DE). Eene bijdrage tot de kennis der kettingbreuken van Stieltjes. Leiden, 1915. Gr. 8vo. 12+102 pp.

WIEN (W.). Die neuere Entwicklung unserer Universitäten und ihre Stellung im deutschen Geistesleben. Rede für den Festakt in der neuen Universität am 29 Juli 1914 zur Feier der hundertjährigen Zugehörigkeit Würzburgs zu Bayern. Leipzig, 1915. M. 1.00

WILLIGENS (C. L.). Transformation de la fonction modulaire $\eta(\omega)$ ainsi que d'une généralization de cette fonction. Genève, 1914. 8vo. 80 pp.

II. ELEMENTARY MATHEMATICS.

AURIC (A.). Quelques théorèmes sur la géométrie du triangle. Paris, Gauthier-Villars, 1915. 8vo. Fr. 1.00

BAILLEUL (—.). See LALANDE (J. DE).

BOHNERT (F.). Ebene und sphärische Trigonometrie. 2te verbesserte Auflage. 2ter Neudruck. Berlin, 1915. M. 2.00

CZEPA (A.). Mathematische Spielereien. (Illustrierte Taschenbücher für die Jugend.) Stuttgart, Union Deutsche Verlagsgesellschaft, 1915. Geb. M. 1.00

DOLINSKI (M.). Algebra und politische Arithmetik. 3te Auflage. Wien, 1915. M. 5.50

HEDRICK (E. R.). Constructive geometry. Exercises in elementary geometric drawing. New York, Macmillan, 1916. Sm. 4to. 8+78 pp. Paper. $0.40

LALANDE (J. DE). Tables de logarithmes étendues à sept décimales par F. C. M. Marie, précédées d'une instruction dans laquelle on fait connaître les limites des erreurs qui peuvent résulter de l'emploi des logarithmes des nombres et des lignes trigonométriques; par A. A. L. Reynaud. Nouvelle édition, augmentée de formules pour la résolution des triangles par Bailleul. Paris, Gauthier-Villars, 1914. 12mo. Cartonné. Fr. 4.00

LEMAN (A.). Vom Dezimalbruch zur Zahlentheorie. (Mathematische Bibliothek, No. 19.) Leipzig, Teubner, 1916. Sm. 8vo. 6+59 pp. Kartonniert. M. 0.80

MARIE (F. C. M.). See LALANDE (J. DE).

MILNE (R. M.). Mathematical papers for admission into the Royal Military Academy and the Royal Military College, September–November, 1915. Edited by R. M. Milne. London, Macmillan, 1916. 30 pp. 1s.

NORRENBERG (J.). Die deutsche höhere Schule nach dem Weltkriege. Beiträge zur Frage der Weiterentwicklung der höheren Schulen gesammelt von J. Norrenberg. Leipzig, Teubner, 1916. Lex. 8vo. 8+275 pp. Geb. M. 5.40

REYNAUD (A. A. L.). See LALANDE (J. DE).

RUSTON (A. G.). Rural arithmetic. London, Clive, 1916. 8vo. 11+431 pp. 3s. 6d.

SANG (E.). A new table of seven-place logarithms of all numbers from 20,000 to 200,000. London, C. and E. Layton, 1916. 18+365 pp. 21s.

WALLACE (E. E.). The circle squared; a solution . . . showing the exact ratio between radius and quadrant. 2d edition. Monmouth, Ill., 1915. 20 pp.

III. APPLIED MATHEMATICS.

AUSTIN (F. E.). Examples in magnetism for students of physics and engineering. 2d edition. Hanover, N. H., 1916. $1.10

BAHRT (W.). Physikalische Messungsmethoden. 2te verbesserte Auflage. Berlin, 1915. M. 0.90

BAIRD (D.). Questions and numerical exercises in physics and chemistry. London, Blackie, 1915. 103 pp. 1s.

BOUSSINESQ (J.). Comment le débit d'un tuyau de conduite, affecté d'un rétrécissement notable, mais graduel, peut se déduire de l'abaissement de pression qui s'y produit le long de la partie rétrécié. Paris, Gauthier-Villars, 1915. 4to. 20 pp. Fr. 1.50

——. Réflexions sur la longue durée de la dynamique rudimentaire d'Aristote et sur son rôle capital jusqu'au jour où fut créée l'analyse infinitésimale. Paris, Gauthier-Villars, 1915. 4to. 16 pp. Fr. 1.00

——. I: Sur le problème du refroidissement de la croûte terrestre, considéré à la manière et suivant les idées de Fourier. II: Calcul correct de l'influence de l'inégalité climatérique sur la vitesse d'accroissement des températures terrestres avec la profondeur sous le sol. Paris, Gauthier-Villars, 1915. 8vo. 34 pp. Fr. 1.50. Another edition, 1915. 8vo. 38 pp. Fr. 1.00

BROWN (E. W.). See DARWIN (G. H.).

CASPARI (C. E.). See ENCYCLOPÉDIE.

CENTNERSZWER (M.). Cours de manipulation de chimie physique et d'électrochimie. Paris, Gauthier-Villars, 1914. 8vo. 8+182 pp. Fr. 6.00

COHN (F.). See ENCYCLOPÉDIE.

DARWIN (F.). See DARWIN (G. H.).

DARWIN (G. H.). Scientific papers. Vol. 5. Supplementary volume containing biographical memoirs. By F. Darwin and E. W. Brown. Lectures on Hill's lunar theory, etc. Edited by F. J. M. Stratton and J. Jackson. Cambridge, University Press, 1916. 55+81 pp. 6s.

DUNOYER (L.). See LORENTZ (H. A.).

DURELL (F.). Fundamental sources of efficiency. Philadelphia, Lippincott, 1914. 8vo. 6+368 pp. Cloth. $2.50

DYCK (W. v.). See KEPLER (J.).

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome V, volume 1, fascicule 1: Histoire des conceptions fondamentales de l'atomistique en chimie d'après l'article allemand de F. W. Hinricksen, Stéréochimie d'après l'article allemand de L. Mamlock; Considérations sur les poids atomiques d'après l'article allemand de E. Study, par M. Joly et J. Roux; Cristallographie, loi fondamentale et son application au calcul et à la représentation des cristaux d'après l'article allemand de T. Liebisch, par F. Walleraut. Leipzig, Teubner, 1915. Gr. 8vo. Pp. 1–96. M. 3.60

——. Tome V, volume 4, fascicule 1: Anciennes théories de l'optique d'après l'article allemand de A. Wangerin, par C. Raveau. Leipzig, Teubner, 1915. Gr. 8vo. Pp. 1–104. M. 3.80

——. Tome VII, volume 1, fascicule 2: Détermination de la longitude et de la latitude d'après l'article allemand de C. W. Wirtz, par G. Fayet; Les horloges, par C. E. Caspari; Théorie des instruments astronomiques de mesures angulaires, des méthodes d'observation et de leurs erreurs d'après l'article allemand de F. Cohn, par J. Mascart. Leipzig, Teubner, 1916. Gr. 8vo. Pp. 225–320. M. 3.60

FAYET (G.). See ENCYCLOPÉDIE.

HIEPE (M.). Die spezifischen Schnittreaktionen des Schubkurbelgetriebes, behandelt nach dem Verfahren von Lagrange. (Diss., Jena.) Weida i. Th., Thomas und Hubert, 1915.

HINRICKSEN (F. W.). See ENCYCLOPÉDIE.

JACKSON (J.). See DARWIN (G. H.).

JOLY (M.). See ENCYCLOPÉDIE.

KEPLER (J.). Nova Kepleriana. Wieder aufgefundene Drucke und Handschriften. Herausgegeben von W. v. Dyck. München, 1915. M. 1.00

KNOBEL (E. B.). See PTOLEMY.

KREUTZ (S.). Elemente der Theorie der Kristallstruktur. Leipzig, 1915. M. 12.00

LIEBISCH (T.). See ENCYCLOPÉDIE.

LOEWY (A.). Versicherungsmathematik. 3te umgearbeitete und vermehrte Auflage. Leipzig, Göschen, 1915. 180 pp.

LORENTZ (H. A.). Les théories statistiques en thermodynamique. Conférences faites au Collège de France en Novembre 1912. Rédigées en 1913 par L. Dunoyer. Leipzig, Teubner, 1916. Gr. 8vo. 4+120 pp. M. 5.80

MAMLOCK (L.). See ENCYCLOPÉDIE.

MASCART (J.). See ENCYCLOPÉDIE.

MONTESSUS (R. DE). Exercices et leçons de mécanique analytique. Paris, Gauthier-Villars, 1915. 8vo. 6+334 pp. Fr. 12.00

PETERS (C. H. F.). See PTOLEMY.

PTOLEMY. Catalogue of stars: a revision of the Almagest. By C. H. F. Peters and E. B. Knobel. Washington, Carnegie Institution, 1916. 3+207 pp.

RAVEAU (C.). See ENCYCLOPÉDIE.

ROUX (J.). See ENCYCLOPÉDIE.

SCHAU (A.). Statik mit Einschluss der Festigkeitslehre. (Aus Natur und Geisteswelt, No. 497.) Leipzig, Teubner, 1915. M. 1.25

STRATTON (F. J. M.). See DARWIN (G. H.).

STUDY (E.). See ENCYCLOPÉDIE.

VALLIER (E.). La ballistique extérieure. 2e édition. Paris, Gauthier-Villars, 1915. 8vo. 212 pp. Cartonné. Fr. 3.00

WALLERAUT (F.). See ENCYCLOPÉDIE.

WANGERIN (A.). See ENCYCLOPÉDIE.

WIRTZ (C. W.). See ENCYCLOPÉDIE.

THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY IN NEW YORK.

The one hundred and eighty-fourth regular meeting of the Society was held in New York City on Saturday, April 29, 1916. The attendance at the morning and afternoon sessions included the following fifty-one members:

Dr. J. W. Alexander, II, Dr. F. W. Beal, Mr. D. R. Belcher, Dr. A. A. Bennett, Professor E. W. Brown, Professor B. H. Camp, Professor C. W. Cobb, Dr. Emily Coddington, Professor F. N. Cole, Professor J. L. Coolidge, Dr. Lennie P. Copeland, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Mr. C. H. Currier, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor C. C. Grove, Dr. Olive C. Hazlett, Professor E. R. Hedrick, Professor E. V. Huntington, Professor Dunham Jackson, Mr. Glenn James, Mr. S. A. Joffe, Professor Edward Kasner, Dr. L. M. Kells, Professor C. J. Keyser, Mr. Harry Langman, Dr. P. H. Linehan, Professor James Maclay, Dr. R. L. Moore, Professor Frank Morley, Mr. G. W. Mullins, Mr. George Paaswell, Professor James Pierpont, Professor H. W. Reddick, Professor R. G. D. Richardson, Mr. J. F. Ritt, Dr. Caroline E. Seely, Dr. H. M. Sheffer, Professor Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Professor Oswald Veblen, Mr. J. H. Weaver, Mr. H. E. Webb, Dr. Mary E. Wells, Professor H. S. White, Mr. J. K. Whittemore, Dr. C. E. Wilder.

The President of the Society, Professor E. W. Brown, occupied the chair, being relieved by Vice-President E. R. Hedrick. The Council announced the election of the following persons to membership in the Society: Dr. E. T. Bell, University of Washington; Professor T. R. Eagles, Howard College; Mr. Glenn James, Purdue University; Dr. J. O. Hassler, Chicago, Ill.; Professor G. N. Watson, University College, London; Mr. J. H. Weaver, West Chester, Pa. Six applications for membership in the Society were received.

Professor D. R. Curtiss was reelected a member of the Editorial Committee of the *Transactions*, to serve for three years beginning October 1, 1916. The resignation of Professor

L. E. Dickson as member of the Editorial Committee was accepted to take effect on October 1, 1916, and Professor L. P. Eisenhart was appointed to fill out the remaining year of Professor Dickson's term. Committees were appointed to prepare a list of nominations of officers and other members of the Council to be elected at the annual meeting, and to consider the matter of the publication of the Harvard Colloquium Lectures. A committee was also appointed, consisting of Professors E. V. Huntington, E. B. Wilson, E. H. Moore, R. C. Archibald, and T. H. Gronwall, to consider in cooperation with other scientific bodies the question of the classification of technical literature.

Thirty-seven members and friends assembled at the dinner after the meeting.

The year 1916 marks the twenty-fifth anniversary of the broadening out of the Society into a national organization, and of the founding of the BULLETIN. It is proposed to celebrate this anniversary in an appropriate manner at the coming summer meeting. Some seventy-five of those who joined the Society in or prior to 1891 have retained their membership during these twenty-five years. It is hoped that a large number of these older members may be present at the summer meeting and that a large representation of the younger generation may also be present to take over the responsibility for the Society's progress during the next quarter century.

The following papers were read at this meeting:

(1) Dr. SAMUEL BEATTY: "Derivation of the complementary theorem from the Riemann-Roch theorem."

(2) Mr. J. F. RITT: "The resolution into partial fractions of the reciprocal of an entire function of genus zero."

(3) Mr. J. F. RITT: "Linear differential equations of infinite order with constant coefficients."

(4) Professor C. J. KEYSER: "Concerning autonomous doctrines and doctrinal functions."

(5) Professor EDWARD KASNER: "Element transformations of space for which normal congruences of curves are invariant."

(6) Mr. J. H. WEAVER: "Some extensions of the work of Pappus and Steiner on tangent circles."

(7) Professor J. L. COOLIDGE: "New definitions for Plücker's numbers."

(8) Professor G. C. EVANS: "Integral equations whose

kernels satisfy a certain difference equation in variable differences."

(9) Professor DUNHAM JACKSON: "An elementary boundary value problem."

(10) Professor L. P. EISENHART: "Transformations of conjugate systems with equal point invariants."

(11) Dr. A. A. BENNETT: "An existence theorem for the solution of a type of real mixed difference equation."

(12) Dr. A. A. BENNETT: "A case of iteration in several variables."

(13) Mr. R. W. BRINK: "Some integral tests for the convergence and divergence of infinite series."

(14) Mr. GLENN JAMES: "A theorem on the non-summability of a certain class of series."

(15) Dr. F. J. MCMACKIN: "Some theorems in the theory of summable divergent series."

(16) Mr. J. R. KLINE: "A definition of sense on plane curves in non-metrical analysis situs."

(17) Professor H. B. FINE: "On approximations to a solution of a system of numerical equations."

(18) Professor B. H. CAMP: "Fourier multiple integrals."

(19) Dr. G. A. PFEIFFER: "On the conformal mapping of curvilinear angles."

(20) Professor G. C. EVANS: "Proof of Green's theorem by approximating polynomials."

(21) Dr. A. R. SCHWEITZER: "On a type of quasi-transitive functional equations."

(22) Dr. J. W. ALEXANDER, II: "Some generalizations of the Jordan theorem."

(23) Dr. C. E. WILDER: "Expansion problems of ordinary linear differential equations with auxiliary conditions at several points."

(24) Professor E. V. HUNTINGTON: "A simple example of the failure of Duhamel's theorem."

(25) Professor W. F. OSGOOD: "Note on functions of several complex variables."

(26) Professor W. A. WILSON: "On separated sets."

Mr. Brink's paper was communicated to the Society by Professor Birkhoff; Dr. McMackin was introduced by Professor Keyser, and Mr. Kline by Dr. R. L. Moore. The papers of Dr. Beatty, Professor Evans, Mr. Brink, Dr. Pfeiffer, Dr. Schweitzer, Dr. Alexander, Professor Osgood, and Professor Wilson were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. By making use of the idea of complementary bases, without which the complementary theorem cannot be even stated, Dr. Beatty derives the complementary theorem from the less general Riemann-Roch theorem.

2. In Mr. Ritt's first paper, which is auxiliary to the second, are discussed the conditions under which the reciprocal of an entire function $G(z)$, of genus zero, can be resolved into simple elements as if $G(z)$ were a polynomial. If the zeros of $G(z)$ are $a_1, a_2, \cdots, a_n, \cdots$, the resolution is possible when, on and after a certain point,

$$\left| \frac{a_{n+1}}{a_n} \right| > 1 + \frac{k}{n},$$

where $k > 2$.

3. The first part of Mr. Ritt's second paper develops the theory of the operator

$$A = \left(1 - \frac{D}{a_1}\right)\left(1 - \frac{D}{a_2}\right) \cdots \left(1 - \frac{D}{a_n}\right) \cdots,$$

where D denotes differentiation and where

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|}$$

is convergent. This operator, which, strangely enough, does not seem to have been discussed before, has the entire space of analytic functions for its domain of applicability. In the second part is discussed the most general solution of the equation $A\varphi(z) = 0$. The solutions are shown to be uniform and to have no isolated singularities. The analytical representation of the solutions is discussed. There is obtained, incidentally, a known result in the theory of analytic prolongation. In the third part of the paper, which will furnish material for future investigations, more general equations are considered, the principal object being to apply the results to the above equation $A\varphi(z) = 0$.

4. The undefined or primitive terms or elements in a postulationally established theory admit of infinitely many interpretations. Hence such terms are variables. Hence the postulates or primitive so-called propositions are not propositions, being neither true nor false prior to specific interpretation, but are propositional functions, and the same holds of the deduced so-called propositions. Accordingly, the theory constituted by such propositional functions, being neither true nor false prior to assignment of specific values to the primitive variables involved, is not a doctrine but is, Professor Keyser contends, something that may be fitly called a doctrinal function. Just as a propositional function is a source of innumerable propositions, some true, some false, the true ones being called values of the propositional function, so a doctrinal function is a source of innumerable doctrines, some true, some false, arising from diverse specific interpretations of the primitive variables. The true doctrines thus arising ought, naturally, to be called the values of the doctrinal function. It thus appears that Hilbert's so-called "Foundations of Geometry," for example, is, strictly speaking, not a geometry nor any other specific doctrine. It is a 3-dimensional euclidean doctrinal function, of which ordinary euclidean geometry is one value, viz., that value that arises from assigning to the primitive variables (Hilbert's "point," "line," and "plane") such meanings as Hilbert did not assign but Euclid did assign (unfortunately under the caption of "definitions" instead of descriptions). No system of primitive propositional functions, or postulates, is satisfied by a set of specific entities that do not have "excessive meaning," i. e., meaning over and above that required merely to satisfy the system. The various doctrines arising from a given doctrinal function are thus known and distinguished by the differing excessive meanings of the entities involved. For a doctrine to be geometric, part of the excessive meaning of the primitive entities must be the concept of spatial extension.

5. Professor Kasner determines all transformations of lineal elements (x, y, z, y', z') of space such that every normal congruence of curves shall be converted into a normal congruence. The infinite group obtained is isomorphic with the group of contact transformations in space. The only transformations in the new group which convert curves into curves

are the conformal transformations, which form a 10-parameter subgroup. The results are of interest in connection with the optics of general isotropic media.

6. Pappus in Book IV of the Collection develops some properties of infinite systems of tangent circles A which are tangent to two given circles. Mr. Weaver's paper extends these results in two directions.

(1) He finds the analytic expression for the radius of the nth circle in the infinite series, and also the expression for certain other allied lines, and from these expressions develops some infinite series which may be summed geometrically. These series closely resemble those given by Fabry,* from an analytic point of view.

(2) He develops a projective method for constructing conics and investigates some projective properties of tangents and normals to the conics determined by the given circle and the series A. The results here are supplementary to those of Steiner on the same problem,† and include additional extensions in the light of modern projective theory.

7. Professor Coolidge's paper gives new definitions for the order, class, and deficiency of an algebraic plane curve suitable to the case where the curve is required to be real.

8. Professor Evans's first paper appears in full in the present number of the BULLETIN.

9. Professor Jackson's paper appeared in full in the May BULLETIN.

10. Professor Eisenhart considers pairs of surfaces S and S_1, so related that for the congruence of the joins of corresponding points M and M_1, the developables meet S and S_1 in conjugate systems of curves. Let S^{-1} and S' denote the Laplace transforms of S with respect to these conjugate curves, taken as parametric. The tangent planes to S osculate one family of parametric curves on S^{-1} and the other family on S'. In each of these osculating planes there is a pencil of conics tangent to the two curves at the points of osculation. These

* Théorie des Séries à Termes constants, Chap. IV.

† Steiner, Werke, herausgegeben von Weierstrass, vol. I, pp. 47 and ff.

conics determine involutions on the line of intersection of the tangent planes to S and S_1. Involutions are likewise determined on these lines by similar pencils of conics in the tangent planes to S_1. These involutions are the same only in case the parametric conjugate systems on S and S_1 have equal point invariants, in which case S and S_1 are in the relation of a transformation K, previously considered by the author. Certain pencils of quadrics of singular interest may be associated with these pencils of conics. The interrelations of these various configurations are determined. The paper will appear in the *Annals of Mathematics*.

11. In this paper Dr. Bennett points out the fact that a real mixed difference equation of the form

$$y_{h+1}^{(0)} = F[x;\ y_0^{(0)},\ y_0^{(1)},\ \cdots,\ y_0^{(k)};$$
$$y_1^{(0)},\ \cdots,\ y_1^{(k_1)};\ \cdots;\ y_h^{(0)},\ \cdots,\ y_h^{(k_h)}],$$

where by $y_i^{(j)}$ is meant $d^jy(x+i)/dx^j$, will under certain simple and general restrictions have as solutions only functions which are continuous together with all of their derivatives for all non-negative values of x. It is then shown that not only are the independent initial conditions infinite in number, but that solutions may be readily constructed having an infinite number of degrees of freedom, where the Taylor's series of the desired solution is assigned arbitrarily, as divergent series if desired, at each of the points $x = i$, $i = 0, 1, \cdots, h$. Use is made of some of the results obtained by Mr. J. F. Ritt in a forthcoming article in the *Annals of Mathematics*. The present paper, also, will appear in the *Annals*.

12. Dr. Bennett here examines a special case of iteration in several variables. The essential equivalence is pointed out between difference equations in several variables and those functional equations which constitute the subject matter of iteration. In particular, the author examines equations of the form $F_{(i)}[u_1, u_2, \cdots, u_n] \equiv au_i + ((u^2))$, $i = 1, 2, \cdots, m$, where by $((u^2))$ is meant a power series in the m variables $(u_1, u_2, \cdots, u_m)$ for which no term appears of lower than the second degree in the set of m variables together. For the case in which $|a| \neq 0, \neq 1$, an explicit solution is obtained in terms of the integral iterates by a formula which is obtained

from an extension of Newton's interpolation formula. The limiting forms are also discussed. This paper constitutes a partial extension of the general theory of one variable contained in a previous paper by the author, *Annals of Mathematics*, volume 17 (1915). This paper will also appear in the *Annals*.

13. The recurrence formula that defines the general term of an infinite series may be regarded as a difference equation in the partial sum of the series. In Mr. Brink's paper methods are given whereby such a difference equation may be replaced by a differential equation whose solution behaves for large integral values of the variable like the partial sum of the series. In this way the author obtains a sequence of integral tests for the convergence and divergence of series. The Maclaurin-Cauchy test is given with extended conditions as the first of this sequence. Another of the simpler tests presented makes use of the function $r(x)$ which has the property that $r(n)$ is the ratio of u_{n+1} to u_n, u_n being the general term of the series. Under certain restrictions upon $r(x)$ the series converges or diverges with the integral

$$\int_a^\infty e^{\int_a^x \log r(x)dx} dx.$$

By means of this and other tests of the sequence a large number of the classical convergence tests are easily established. The tests are readily generalized for multiple series.

14. Mr. James extends the notion of proper divergence to include a certain class of oscillating series. This class contains all series such that for any positive (or negative) C and every M there exists an $m \geqq M$ such that

$$\sum_{p=1}^{n} (S_{m+p} - C) \geqq 0, \text{ (or } \leqq 0), \ n \geqq m + 1.$$

A theorem is established from which it follows that the methods of Borel, Cesàro, LeRoy, Cesàro-Riesz, and others cannot sum series of this class.

15. Hardy and Smail showed that the sufficient condition for the continuity of the sum function of a summable divergent

series of continuous functions is the uniform summability of the series, that is, the uniform convergence of the auxiliary limit, as Borel's integral, Cesàro's mean value, etc. Dr. McMackin shows that this is not a necessary condition for the continuity of the sum, and derives conditions which are both necessary and sufficient. For series which oscillate between finite limits quasi-uniform convergence of the auxiliary limit is shown to be both necessary and sufficient in the case of Borel's integral sum, while in the other case it holds for all summable series with continuous terms. It is also shown that a quasi-uniformly convergent series is quasi-uniformly summable according to the definition of quasi-uniform summability given.

16. Mr. Kline proposes the following definition for sameness of sense on two simple closed curves: The sense $A_1B_1C_1$ on the simple closed curve J_1 and the sense $A_2B_2C_2$ on the simple closed curve J_2 are said to be the same with respect to their common exterior E_{12}, if there exist a simple closed curve J_3 lying entirely in E_{12} and three points A_3, B_3, C_3 on J_3 such that (1) if it is impossible to join A_1 to A_3, B_1 to B_3 and C_1 to C_3 by simple continuous arcs which except for their end points lie entirely in E_{13} (the common exterior of J_1 and J_3) and have no points in common, then it is also impossible to join A_2 to A_3, B_2 to B_3 and C_2 to C_3 by simple continuous arcs lying except for their end points entirely in E_{23} (the common exterior of J_2 and J_3) and having no points in common, or (2) if it is possible to join A_1 to A_3, B_1 to B_3, and C_1 to C_3 in the manner above indicated, then it is also possible to join A_2 to A_3, B_2 to B_3 and C_2 to C_3 as indicated.

It is established that if for one choice of the curve J_3 and of three points A_3, B_3, and C_3 lying thereon, the senses $A_1B_1C_1$ and $A_2B_2C_2$ are the same with respect to E_{12}, then these senses are also the same with respect to any other such choice of J_3 and $A_3B_3C_3$.

It is also shown that if the senses $A_1B_1C_1$ on J_1 and $A_2B_2C_2$ on J_2 are opposite with respect to their common exterior and the senses $A_2B_2C_2$ on J_2 and $A_3B_3C_3$ on J_3 are opposite with respect to their common exterior, then the senses $A_1B_1C_1$ on J_1 and $A_3B_3C_3$ on J_3 are the same with respect to their common exterior.

17. Professor Fine's paper is concerned with the proof of the following theorem: Let $f_i(x_1, x_2, \cdots, x_n)$, $i = 1, 2, \cdots, n$, be real functions of the real variables $x_1, x_2, \cdots, x_n$ which have continuous first and second derivatives in the region under consideration. Let $(x_1^0, x_2^0, \cdots, x_n^0)$ belong to this region and let $\xi_1, \xi_2, \cdots, \xi_n$ be the numbers defined by the equations

$$f_i(x_1^0, x_2^0, \cdots, x_n^0) + \sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k^0} \xi_k = 0 \quad (i = 1, 2, \cdots, n).$$

Again let S denote the circle, sphere, or hypersphere whose center is $(x_1^0 + \xi_1, x_2^0 + \xi_2, \cdots, x_n^0 + \xi_n)$ and whose radius is $\rho = [\Sigma \xi_k^2]^{\frac{1}{2}}$, and suppose that, for this region S, $D > 0$ denotes the lower bound of the absolute values of the functional determinant of the functions f_i, $M < \infty$ the upper bound of the absolute values of the first minors of this determinant, and $N < \infty$ the upper bound of the absolute values of the second derivatives of the functions f_i.

Then if

$$\left[\sum_{i=1}^{n} f_i^2(x_1^0, x_2^0, \cdots, x_n^0) \right]^{\frac{1}{2}} < \frac{D^2}{n^3 \sqrt{n} M^2 N},$$

the system of equations $f_i(x_1, x_2, \cdots, x_n) = 0$ has one and but one solution in S, and this solution may be approximated to uninterruptedly by successive determinations of $\xi_1^{(j)}, \xi_2^{(j)}, \cdots, \xi_n^{(j)}$ and $x_1^{(j+1)}, x_2^{(j+1)}, \cdots, x_n^{(j+1)}$ by the formulas

$$f_i(x_1^{(j)}, x_2^{(j)}, \cdots, x_n^{(j)}) + \sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k^{(j)}} \xi_k^{(j)} = 0,$$

$$x_i^{(j+1)} = x_i^{(j)} + \xi_i^{(j)},$$

the solution being $\lim_{j \to \infty} (x_1^{(j)}, x_2^{(j)}, \cdots, x_n^{(j)})$.

18. By using G. H. Hardy's definition of a non-absolutely convergent multiple integral it is possible to show that an arbitrary function may be expressed as a Fourier double integral under circumstances much more general than is possible when the iterated integral is used. The results are useful in connection with the Fourier quadruple integral which appears in three dimensional physical problems. Professor Camp's paper also considers the continuity, differentiability,

and integrability with respect to parameters of multiple integrals over infinite fields.

19. In a previous paper Dr. Pfeiffer has shown that divergent power series may be formally obtained in seeking a conformal transformation which maps a curvilinear angle upon a rectilinear angle of the same magnitude and which is analytic about the vertex of the angle. In the present paper he shows that there exist functions $f(z)$ which map the interior of the curvilinear angle upon the interior of a rectilinear angle of the same magnitude conformally such that the sum of the first n terms of the power series already referred to represents the function $f(z)$ asymptotically up to order n. The mapping defined by the function $f(z)$ is conformal on the sides of the angle, except of course at the vertex, where it is continuous.

The above refers to angles whose magnitudes are not commensurable with π, for in the contrary case the writer has already shown that all transformations (analytic about the vertex) obtained formally are convergent.

20. Many problems in mathematical physics are expressed in terms of integral relations, and it is only by passing to a limit that the differential equations are secured. There may be nothing in the physical problem which corresponds to the limit. An analysis is therefore desirable which involves rather the integral than the differential relations. Several proofs have been given of Green's theorem on this basis. Professor Evans notes a new proof, possible by means of an expansion in polynomials, of the same nature as the proof of a theorem in de la Vallée Poussin's Cours d'Analyse, volume II, page 24, edition of 1912.

21. Let $\lambda_i(x)$ $(i = 0, 1, 2, \cdots, n+1)$ be functions of a single variable; then Dr. Schweitzer defines the type of quasi-transitive functional equations

$$f\{\lambda_1 u_1, \lambda_2 u_2, \cdots, \lambda_{n+1} u_{n+1}\} = \lambda_0 f\{\phi_1, \phi_2, \cdots, \phi_{n+1}\},$$

where $u_j = f(t_1, t_2, \cdots, t_n, x_j)$, $j = 1, 2, \cdots, n+1$, and ϕ_j denote functions of the variables $x_1, x_2, \cdots, x_{n+1}$. In the present paper the special case $\phi_j = \sum_{k=1}^{n+1} m_{jk} x_k$ only is considered. Then the resulting class of functional equations may be put

into (1, 1) correspondence with the class of linear homogeneous substitutions and one readily defines, at least formally, a representation of an arbitrary finite group of such substitutions. On the other hand, if $\lambda_i(x) = x$ $(i = 0, 1, 2, \cdots, n+1)$ and if $f\{x_1 + y_1, \cdots, x_{n+1} + y_{n+1}\} = f(x_1, \cdots, x_{n+1}) + f(y_1, \cdots, y_{n+1})$, i. e., if $f(x_1, x_2, \cdots, x_{n+1}) = \sum_{j=1}^{n+1} \xi_j \cdot x_j$, then one obtains $n+2$ equations conditioning the constants ξ_j. By rational elimination, it is found that each of the constants $\xi_2, \xi_3, \cdots, \xi_{n+1}$ satisfies an algebraic equation of the nth degree with coefficients which are rational functions of the constants $M_{st} = m_{st} - m_{1t}$ $(s, t = 1, 2, 3, \cdots, n+1; s_1 \neq 1)$. In particular, ξ_{n+1} is a root of the characteristic equation of a certain linear homogeneous substitution.

22. Dr. Alexander's paper contains a simplified proof of Jordan's theorem that a simply closed curve subdivides the plane into two and only two regions. By generalizing the method employed, theorems analogous to Jordan's theorem about k-dimensional manifolds in n-dimensional space are proved.

23. In a paper in the *Transactions* for 1908 Birkhoff considers certain problems connected with the differential system consisting of the equation

$$\frac{d^n u}{dx^n} + * + p_2 \frac{d^{n-2} u}{dx^{n-2}} + \cdots + p_n u + \lambda u = 0$$

and n boundary conditions $W_i(u) = 0$, in which the W's are linear combinations of the values of u and its first $n-1$ derivatives at the end points of the interval over which the system is considered. Among other things he defines a Green's function, $G(x, s, \lambda)$, for this system and proves that for a system with "regular" boundary conditions

$$\lim_{m=\infty} \frac{1}{2\pi i} \int_\Gamma \int_a^b G(x, s, \lambda) f(s) ds d\lambda = f(x),$$

for any function $f(x)$ possessing a continuous derivative, where Γ is a contour in the λ plane enclosing the first m poles of $G(x, s, \lambda)$.

In the present paper Dr. Wilder extends this result in that

he adds to the W's linear combinations of the values of u and its first $n-1$ derivatives at any finite number of points interior to the interval. The Green's function for this system is then defined by the same formula as is used by Birkhoff and it is found that the above integral converges to $f(x)$ provided $f(x)$ has a certain number of derivatives, which number never need exceed n, and provided certain determinants formed from the constants of the auxiliary conditions do not vanish. In the case n is even it is further necessary to assume that the second point from either end of the interval is farther from that end than the first point from the other end is from that end.

24. Professor Huntington's paper refers to the theorem of Duhamel already discussed by Osgood, R. L. Moore, and Bliss in the *Annals of Mathematics* for 1903, 1912, and 1914, namely: If $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\beta_1, \beta_2, \cdots, \beta_n$ are two sets of infinitesimals such that $\lim\limits_{n=\infty} (\beta_i/\alpha_i) = 1$; and if $\lim\limits_{n=\infty} [\alpha_1 + \alpha_2 + \cdots + \alpha_n] = a$ exists, then $\lim\limits_{n=\infty} [\beta_1 + \beta_2 + \cdots + \beta_n]$ will also exist, and be equal to a. The following example of the failure of this theorem is simpler than examples that have been previously given: Let $\alpha_i = a/n$, and $\beta_i = a/n + 2ic/n^2$, where a and c are fixed constants. Then $\lim \beta_i/\alpha_i = 1$ as $n = \infty$; but $\lim \Sigma\alpha_i = a$, while $\lim \Sigma\beta_i = a + c$.

25. Professor Osgood's paper appeared in full in the June BULLETIN.

26. Professor Wilson's paper appeared in full in the May BULLETIN.

F. N. COLE,
Secretary.

APPLICATION OF AN EQUATION IN VARIABLE DIFFERENCES TO INTEGRAL EQUATIONS.

BY PROFESSOR G. C. EVANS.

(Read before the American Mathematical Society, April 29, 1916.)

IT is known that if the kernel of an integral equation of Volterra type is in the simple form of the difference alone of the

two variables, then the kernel of the resolvent equation is in the same form. We shall see presently that this result also holds for the kernel of the Fredholm equation, provided that it is periodic, the period being equal to the interval of integration. Moreover, the same condition of periodicity is sufficient to make the kernel of the resolvent equation of the same form as the original kernel when the latter is expressed by the formula

$$K(x, y) = \Psi(x + y) + \Theta(x - y). \tag{1}$$

An obvious way to approach the problem is to make use of the partial differential equations

$$\frac{\partial K}{\partial x} + \frac{\partial K}{\partial y} = 0 \quad \text{and} \quad \frac{\partial^2 K}{\partial x^2} = \frac{\partial^2 K}{\partial y^2}, \tag{2}$$

which say respectively that the kernel $K(x, y)$ is a function of the difference of the two variables, or a function of the form (1). The objection to this method is that there is nothing in the theory of the Fredholm equation which demands the existence of derivatives, and nothing which directly refers to them in the statement of the problem.

Less immediate, and also less restrictive, is a treatment by means of Fourier series. The coefficients in the trigonometric development of the resolvent kernel have simple expressions in terms of those of the given kernel, on account of the special form of the latter. One can, in fact, get an instructive aperçu of the facts in the general problem of the Fredholm equation by considering, with these elementary methods, this special case. Here again, however, more seems to be demanded in the nature of the kernel than should be called for by the question which is the subject of this paper.

§ 1. *Some Difference Equations in Variable Differences.*

1. A necessary and sufficient condition that a function $F(x, y)$ be in the form $f(x + y)$ is that it admit the substitution $x' = x + t$, $y' = y - t$, where t is arbitrary. Likewise, a necessary and sufficient condition that a function be in the form $\varphi(x - y)$ is that it admit the substitution $x' = x + t$, $y' = y + t$. These conditions may be respectively expressed by means of the difference equations

$$F(x+t,\ y-t) - F(x,\ y) = 0, \tag{3}$$

$$F(x+t,\ y+t) - F(x,\ y) = 0. \tag{4}$$

2. In order to distinguish the more interesting case where $K(x, y)$ is of the form (1), let us assume at first that $K(x, y)$ is defined everywhere in the plane except at a point set of measure zero.* *A necessary and sufficient condition that we may write* $K(x, y)$ *in the form* (1), *except possibly at a set of points of measure zero, is that the condition*

$$\begin{aligned} K(x+t+t', y+t-t') &- K(x+t, y+t) \\ &- K(x+t',\ y-t') + K(x,\ y) = 0 \end{aligned} \tag{5}$$

shall hold for all values of x, y, t, t', *except possibly for a point set of zero measure, one of the four points* (x, y), $(x+t,\ y+t)$, $(x+t',\ y-t')$, $(x+t+t',\ y+t-t')$ *lying in that set.*

3. The relation (5) says merely that if we take a rectangle of which the sides are parallel to the lines $x+y=0$ and $x-y=0$, the sum of the values of the function at the extremities of one diagonal is equal to the sum of its values at the extremities of the other, i. e., if 1, 2, 3, 4 are the vertices in cyclic order, we have the equation

$$K_1 + K_3 = K_2 + K_4. \tag{5'}$$

If we have a second rectangle 3, 4, 5, 6 which has a side in common with the first, it is seen that *in the relations*

$$\begin{aligned} K_1 + K_3 &= K_2 + K_4, \\ K_3 + K_5 &= K_4 + K_6, \\ K_1 + K_5 &= K_2 + K_6, \end{aligned} \tag{6}$$

the first two imply the third.

Let us define a null set L as made up of the following one-dimensional sets. Let it include all the lines $x+y=$ const. or $x-y=$ const. on which there is a not-null set of points where $K(x, y)$ is not defined, and let x_0, y_0 be a point where

---

* The discussion of this problem in connection with equation (5), below, as applied to point sets in general, offers an interesting generalization of some aspects of the theory of the hyperbolic differential equation: in particular, direct proofs of some of the existence theorems. (See a paper by the author soon to be published.

$K(x, y)$ still remains defined. On the line $x + y = x_0 + y_0$ there lie at most a null set of points where $K(x, y)$ is not defined. Include in L the lines $x - y =$ const. which go through these points; and operate similarly on the line $x - y = x_0 - y_0$. The total set formed in this way constitutes L.

The theorem of Section 2 will be established if we show that *a necessary and sufficient condition that $K(x, y)$ be in the form* (1), *except in L, is that the relation* (5) *hold unless one of the four points mentioned lie in L.*

4. That the condition is necessary follows at once by direct substitution. In order to show that it is also sufficient, let us write $K(x, y)$ in the form $\Phi(x + y, x - y)$, and consider (5) in the form

$$\Phi(x + y + 2t, x - y + 2t') - \Phi(x + y + 2t, x - y)$$
$$= \Phi(x + y, x - y + 2t') - \Phi(x + y, x - y),$$

which tells us that the right-hand member is invariant of the substitution $x' = x + t$, $y' = y + t$. We can give t then such a value that $x + y + 2t = x_0 + y_0$, and write

$$\Phi(x_0 + y_0, x - y + 2t') - \Phi(x_0 + y_0, x - y)$$
$$= \Phi(x + y, x - y + 2t') - \Phi(x + y, x - y)$$

or

$$\Phi(x + y, x - y + 2t') - \Phi(x_0 + y_0, x - y + 2t')$$
$$= \Phi(x + y, x - y) - \Phi(x_0 + y_0, x - y).$$

The right-hand member of this equation is thus seen to be invariant of the substitution $x' = x + t'$, $y' = y - t'$, wherever it is defined, and so we can give to t' a value such that $x - y + 2t' = x_0 - y_0$. Hence we have

$$\Phi(x + y, x_0 - y_0) - \Phi(x_0 + y_0, x_0 - y_0)$$
$$= \Phi(x + y, x - y) - \Phi(x_0 + y_0, x - y)$$

or

$$(7)\quad K(x, y) = \Phi(x + y, x_0 - y_0) + \Phi(x_0 + y_0, x - y)$$
$$- \Phi(x_0 + y_0, x_0 - y_0)$$

which proves the theorem. This result can also be obtained by a more geometrical treatment.

5. If instead of being defined over the entire plane, the function $K(x, y)$ is defined over the rectangle R: $a < x < A$, $b < y < B$, it may be extended over the entire plane by first defining it in the interior of the circumscribed rectangle with sides parallel to the lines $x + y = 0$ and $x - y = 0$, and then defining it as having the value zero on the boundary of this rectangle and over the rest of the plane. The definition in this circumscribed rectangle is established by means of the relation (5′), and its uniqueness follows almost immediately by means of the relation (6). By then applying the theorem of Section 2 we have the following

COROLLARY 1. *The theorem of Section 2 holds, if instead of being defined over the entire plane, except for a point set of measure zero, it is defined, with the same exception, over the rectangle R: $a < x < A$, $b < y < B$.*

We have also the following theorems:

COROLLARY 2. *The condition* (5) *will still be necessary and sufficient if we add the restriction that $|t|$ and $|t'|$ are to be less than η, where η is assigned arbitrarily, positive, dependent on x, y if desired.*

This theorem is deduced immediately with the help of equation (6).

COROLLARY 3. *If the function $K(x, y)$ is continuous at the points in which it is defined, the condition* (5) *may be replaced by one in which $t' = t$, i. e., by the condition*

$$(8) \quad K(x + 2t, y) - K(x + t, y + t) - K(x + t, y - t) + K(x, y) = 0.$$

In fact, from (8) it follows that

$$K(x + mt + nt, y + mt - nt) - K(x + mt, y + mt) - K(x + nt, y - nt) + K(x, y) = 0,$$

where m and n are integers, and since every point in R is a point of the form $(x + mt + nt, y + mt - nt)$ or a limiting point of such points, we have the relation (5).

COROLLARY 4. *The relation* (5) *may be replaced by the one obtained from it by interchanging t and t', or by both relations together, and the theorem will still hold.*

COROLLARY 5. *A necessary and sufficient condition that we may write $K(x, y)$ in the form*

$$K(x, y) = f(x) + g(y) \tag{9}$$

is that

$$K(x+t, y+t') - K(x, y+t') - K(x+t, y) + K(x, y) = 0. \tag{9'}$$

§ 2. *Periodicity.*

6. The kernel of an integral equation and the kernel of the resolvent equation are connected by Volterra's relation. If we are dealing with the equation of Fredholm type containing a parameter λ, the relation has the well known form

$$K(x, y) + k_\lambda(x, y) = \lambda \int_a^b K(x, \xi) k_\lambda(\xi, y) d\xi = \lambda \int_a^b k_\lambda(x, \xi) K(\xi, y) d\xi. \tag{10}$$

For generality the integral may be taken in the Lebesgue sense. From (10) it follows obviously that if $K(x, y)$ is periodic in x with a certain period, $k_\lambda(x, y)$ is periodic in x with the same period; and similarly for y.

In regard to the kernels of the form (1), in which we are more specially interested, *a necessary and sufficient condition that $K(x, y)$ be periodic, with period p, in both arguments, is that $\Psi(x + y)$ and $\Theta(x - y)$, as functions of a single variable, be each periodic with the same period p.* Obviously the condition is sufficient. To show that it is necessary, notice that we have

$$K(x+t, y+t) - K(x, y) = \Psi(x+y+2t) - \Psi(x+y),$$

whence, taking $t = p$,

$$\Psi(x+y+2p) = \Psi(x+y). \tag{11}$$

But also

$$K(x+p+t, y+t) - K(x+p, y) = \Psi(x+y+p+2t) - \Psi(x+y+p),$$

so that

$$\Psi(x+y+p+2t) - \Psi(x+y+p) = \Psi(x+y+2t) - \Psi(x+y),$$

and if $t = \frac{1}{2}p$,

$$\Psi(x + y + 2p) + \Psi(x + y) = 2\Psi(x + y + p).$$

Comparing this with (11), we have

$$\Psi(x + y + p) = \Psi(x + y).$$

From this and the periodicity of $K(x, y)$ it follows that

$$\Theta(x - y + p) = \Theta(x - y).$$

Nothing is changed in these theorems if we except a point set of zero measure (in the case of the second theorem, a point set of the type L).

§ 3. *Kernels of the Form* $\Psi(x + y) + \Theta(x - y)$.

7. Let us understand the integrals to be taken in the Lebesgue sense, and the function $K(x, y)$ to be summable, with its square, over any two-dimensional region. The Volterra relations (10) hold with this assumption except at a point set of measure zero, and by giving a complete definition of the function $K(x, y)$ they can be made to define a function $k(x, y)$ at every point, finite or infinite, as the case may be. We have the following theorem:

THEOREM 1. *If, except at the points of a set L (see Section 3) of measure zero, the kernel $K(x, y)$ can be written in the form (1), in which the functions Ψ and Θ, as functions of a single argument, are periodic with period $b - a$, then with the same exception, the resolvent kernel $k(x, y)$ can be written in the same form, and has the same properties of periodicity.*

As has already been noticed in Section 6, it is the same thing as above to assume that $K(x, y)$ is periodic in each argument with period $b - a$, and the periodicity of $k(x, y)$ follows from it. By virtue of the assumption of periodicity in regard to $K(x, y)$ the point set L is itself periodic.

8. In order to prove the theorem let us perform on the second of the equations (10) the operations indicated in (5), and also the operation obtained from this by interchanging t and t'. Let us denote the result of performing these operations on a function $f(x, y)$ by $H_f(t, t'|x, y)$ and $H_f(t', t|x, y)$ respectively. In forming the resulting pair of equations we have to perform such reductions as the following:

$$\int_a^b k_\lambda(x + t + t', \xi)\Psi(\xi + y + t - t')d\xi$$
$$= \int_{a+t-t'}^{b+t-t'} k_\lambda(x + t + t', \xi - t + t')\Psi(\xi + y)d\xi,$$

which reduces to

$$\int_a^b k_\lambda(x + t + t', \xi - t + t')\Psi(\xi + y)d\xi$$

on account of the periodicity.* Hence we obtain finally the pair of simultaneous integral equations:

$$
\begin{aligned}
&H_K(t, t'|x, y) + H_k(t, t'|x, y) \\
&\qquad = \lambda\left[\int_a^b H_k(t', t|x, \xi)\Psi(\xi + y)d\xi\right. \\
&\qquad\qquad \left. + \int_a^b H_k(t, t'|x, \xi)\Theta(\xi - y)d\xi\right], \\
(12)\qquad &H_K(t', t|x, y) + H_k(t', t|x, y) \\
&\qquad = \lambda\left[\int_a^b H_k(t', t|x, \xi)\Theta(\xi - y)d\xi\right. \\
&\qquad\qquad \left. + \int_a^b H_k(t, t'|x, \xi)\Psi(\xi + y)d\xi\right].
\end{aligned}
$$

In this system of equations the kernels are the functions Ψ and Θ, the known functions are the $- H_K$ and the solutions desired are the functions H_k. It is worth noticing that since the kernels are independent of t and t', any of the functions belonging to $K(x, y)$ gives rise to a solution of the homogeneous system of equations corresponding to (12), and vice versa; hence the characteristic values of λ in (12) are merely the roots of $K(x, y)$. In fact if we write $t = t'$ in (12), the pair of equations reduce to a single equation of which the kernel is $K(x, y)$. The solutions of (12) will then be determined unless λ happens to be root of $K(x, y)$.

Since for given values of t and t' the known functions vanish except at the points of a set of zero measure (i. e., except when

* In the treatment by means of the partial differential equation, this change of variable corresponds to an integration by parts.

one of the seven points (x, y), $(x + t, y + t)$, $(x + t, y - t)$, $(x + t', y + t')$, $(x + t', y - t')$, $(x + t + t', y + t - t')$, $(x + t + t', y - t + t')$ lie in L), the solutions will vanish except at a point set of zero measure. Moreover from the formal character of the solution this set will not contain not-null sets of points on any line parallel to the X or Y axes. Hence the right hand members of (12) will vanish identically, and at all points, even at points of L, we shall have the result

$$H_k(t, t'|x, y) = - H_K(t, t'|x, y), \tag{13}$$

which asserts that $H_k(t, t'|x, y)$ and $H_k(t', t|x, y)$ both vanish for all values of x, y, t, t' such that none of the seven points before mentioned lie in L. Hence, by means of Corollary 4 in Section 5, $k(x, y)$ will have the form (1), except for points of L; and the theorem is proved.

9. If instead of being defined over the entire plane, the kernel $K(x, y)$ is defined only for values of x and y between a and b, the theorem may be extended to cover this case. For if the kernel is in the form (1), it is defined thereby over the interior of the circumscribing rectangle mentioned in Section 5. In fact, that is the region of variation for x and y determined by the arguments $x + y$ and $x - y$, on which depend the functions Ψ and Θ. In order to complete the definition of the function over the entire plane in the manner best to fit with Theorem 1, let us cover the plane by repeating the circumscribing rectangle with the attached values of $K(x, y)$, thus making $K(x, y)$ periodic in x and y separately with period $2(b - a)$. By means of Theorem 1, then, we have the following theorem:

THEOREM 2. *If for values of x and y between a and b the kernel $K(x, y)$ can be written in the form* (1), *in which the functions Ψ and Θ, as functions of a single argument, are periodic with period $b - a$, the hypothesis holding except at a point set of zero measure, of the form L (see Section* 3), *then with the same exception, the resolvent kernel $k(x, y)$ can be written in the same form, and has the same properties of periodicity.*

10. The corresponding theorem is true for functions of the form

$$K(x, y) = \Theta(x - y). \tag{14}$$

In fact, if we make use of the equation

$$K(x+t,\, y+t) - K(x,\, y) = 0, \tag{15}$$

we can deduce by means of it, in the same way as before we used the relation (5), the following theorem:

THEOREM 3. *Theorems* 1 *and* 2 *hold if instead of considering kernels of the form* (1) *we consider kernels of the form* (14).

The corresponding theorem is not true for kernels which are functions of $x + y$ alone. In regard to functions of the form (14), however, we may go still further, if they are continuous.

THEOREM 4. *If $K(x, y)$ is in the form* (14) *and is continuous, a necessary and sufficient condition that $k(x, y)$ should be in the same form is that Θ, as a function of a single argument, be periodic with period $b - a$.*

It may be noticed that if the kernel is a function of x alone, or is a function of y alone, the resolvent kernel is of the same form; but if the kernel is of the form (9), that is, a function of x alone plus a function of y alone, the resolvent kernel is not necessarily of the same form. In fact, if $f_1(x)$ is the resolvent associated with $f(x)$, and $g_1(y)$ is the resolvent associated with $g(y)$, then the resolvent associated with $f(x) + g(y) - \int_a^b f(\xi)g(\xi)d\xi$ is $f_1(x) + g_1(y) - f_1(x)g_1(y)(b - a)$.*

§ 4. *Applications.*

11. On account of the fact that kernels of the form (1) need not be continuous, they may often be used as approximations for particular kernels which are not themselves in that form. The resolvent kernel will then again under the hypothesis of Theorem 1 be in the form (1), and the method of development in trigonometric functions of period $b - a$ yield immediate results. Indeed, functions of the form (1) are determined if they are known along a line $x + y = c_1$ and a line $x - y = c_2$; a remark which simplifies especially the treatment of the integral equation of Volterra type, since it leaves only the single function φ to be calculated. The subject of approximation has not been much studied in connection with integral equations.

---

* See Section 12 below, also Evans, "L'algebra delle funzioni permutabili e non permutabili," *Rendiconti del Circolo Matematico di Palermo*, vol. 34, p. 7.

12. If $K_1(x, y)$ and $K_2(x, y)$ are functions of the form (1), then it is easily shown by means of (5) that the integral combination

$$(16) \qquad \int_a^b K_1(x, \xi)K_2(\xi, y)d\xi$$

is of the same form, provided that the hypothesis of periodicity (the period being $b - a$) holds for the parts Ψ_1, Θ_1 and Ψ_2, Θ_2 of K_1 and K_2. This is the integral combination which has been so extensively studied by Volterra.

Now it is known that if we are given two functions K_1 and K_2 of x and y, and their respective resolvent kernels k_1 and k_2, there may be built up out of them by means of the combination (16) a new kernel and its resolvent; in fact, the function

$$(17) \qquad k_1(x, y) + k_2(x, y) - \int_a^b k_1(x, \xi)k_2(\xi, y)d\xi$$

is resolvent for the function

$$(18) \qquad K_2(x, y) + K_1(x, y) - \int_a^b K_2(x, \xi)K_1(\xi, y)d\xi.*$$

We have then the theorem:

If we have $K_1(x, y) = \Psi_1(x + y) + \Theta_1(x - y)$ *and* $K_2(x, y) = \Psi_2(x + y) + \Theta_2(x - y)$, *where* Ψ_1, Θ_1 *and* Ψ_2, Θ_2, *as functions of a single argument, are periodic with period* $b - a$, *and if we denote by* $k_1(x, y)$ *the function resolvent to* $K_1(x, y)$, *and by* $k_2(x, y)$ *the function resolvent to* $K_2(x, y)$, *then the functions given by* (17) *and* (18) *are of the same form, have the same properties of periodicity, and are themselves mutually resolvent kernels.*

RICE INSTITUTE,
April, 1916.

OPERATORS IN VECTOR ANALYSIS.

BY DR. VINCENT C. POOR.

IN a note in the April BULLETIN on "Changing surface to volume integrals," Professor E. B. Wilson asks why my paper in the January BULLETIN was not made shorter by using the Gibbs-Wilson notation. While the brevity and suggestiveness

* See the footnote to Section 10. For purposes of symmetry and convenience of statemen we have taken $\lambda = 1$ and assumed it not to be a root of K_1 or K_2.

of the Gibbs-Wilson notation is admitted, the note is misleading. For had brevity been the chief aim of my paper the notation of Burali-Forti and Marcolongo could have been made to compare very favorably with Professor Wilson's compact reproduction of the formulas in the Gibbs notation. (Compare the analytic statement of the theorems in the two notations.) However, as one of the purposes of my paper was to exhibit the operational feature of the system of Burali-Forti and Marcolongo, obviously the Gibbs-Wilson notation did not lend itself to this end. Moreover, since the notation of Burali-Forti and Marcolongo is not so well known in this country, some explanation seemed to be necessary.

Professor Wilson says in closing: "The use of words like grad, div, rot is hampering: we no longer write Cubus $\bar{m}$ Census $\bar{p}$ 16 rebus aequatur 40 for $x^3 - 8x^2 - 16 = 40$." Everybody admits the last part of this statement. But we still use for particular kinds of functions or operators such symbols as log, sin, cos, etc., arcsin, etc., sinh, cosh, etc. That the use of such "words" as grad, div, rot is hampering, seems to be a matter of opinion, since they may be used interchangeably with other symbols in both notations.

It is unfortunate that Professor Wilson introduced cartesian coordinates into his proof, since a coordinate system has no proper place in vector analysis. But this seems to be characteristic of the Gibbs-Wilson system. In fact Burali-Forti and Marcolongo have pointed out how the dyadics of Gibbs constantly depend on cartesian coordinates,* a nonlinear system.

UNIVERSITY OF MICHIGAN.

SHORTER NOTICES.

Grundlehren der Mathematik. Der zweite Band des ersten Teils: *Algebra.* By EUGEN NETTO. Leipzig, Teubner, 1915. xii+232 pp.

THE Grundlehren der Mathematik, für Studierende und Lehrer is a series of four volumes on the elements of mathematics appearing from the press of B. G. Teubner under joint authorship as follows: Part I (two volumes), Die Grundlehren

* Burali-Forti et Marcolongo, Transformations linéaires, 1912, p. 147.

der Arithmetik und Algebra, by E. Netto and C. Färber; Part II (two volumes), Die Grundlehren der Geometrie, by W. Fr. Meyer and H. Thieme. One of the objects of this series is to present the fundamentals of mathematics with particular regard to the influences which a century's advances in the science, especially in the direction of greater accuracy and completeness, have had upon its elementary phases. In addition it is proposed to present in the second volume of each part extensions which will serve as an introduction to enquiries which a deeper understanding of elementary mathematics renders feasible. There is, therefore, an interest attached to the selection of topics which the authors have made, and the volume Algebra, by Dr. Netto, is in this way noteworthy. We find in this algebra nothing about convergency of series. A topic affording choice between two methods of treatment, one involving analytic functions, as for instance the theory of the roots of unity, is consistently developed by the algebraical method. Certain arithmetical topics usually found in algebras, such as permutations and combinations, are also omitted. On the other hand, there is given a rather complete treatment of differentiation of rational functions, including topics in maxima and minima, two proofs of the fundamental theorem of algebra, a chapter on cyclotomy, and a proof of the Abel-Ruffini theorem.

The developments in the text follow an elementary mode, and contain numerous particular examples solved out. In the first chapter determinants are treated as summations, use being made, in determination of the signs of the terms, of -1 to the power $[\alpha_1\alpha_2 \cdots \alpha_n]$, where the latter symbol stands for the number of transpositions from the order $1, 2, \cdots, n$ afforded by the permutation $\alpha_1, \alpha_2, \cdots, \alpha_n$. With the aid of the properties of this symbol the standard theorems are developed as theorems on summations. The second chapter, on rational functions, discusses and illustrates definitions, graphical representations, continuity and limits, and derivatives are introduced by means of the expansion of a polynomial $f(x + h)$ in powers of h. Among the examples of maxima and minima of rational functions occurs that of the function $f(x) = (a - x)^\alpha (b - x)^\beta$, $(a < b)$, where α, β are parameters representing only positive integers > 1. The special value of such a problem is that it separates into a definite number of cases (four), according to the possibilities (α even, β even), (α even, β odd), etc.

Such a problem, requiring separation into cases, affords to the student a real illustration of scientific method.

Chapter three on integral functions takes up constructions by Lagrange's method of interpolation, Euclid's algorithm, partial fractions, and reducibility. A trial method of resolving a polynomial with integral coefficients into factors with integral coefficients is next discussed and illustrated. There follow Eisenstein's and Gauss's theorems on this problem, the former finding application in Chapter ten in a proof of the irreducibility of $(x^p - 1)/(x - 1)$ (p a prime number). There is then given a discussion of a domain of rationality and functions in a domain. After a brief discussion in Chapter four of elementary properties of equations, Chapter five gives the theory of linear dependence and the practice of solution, both approximate and exact, of systems of linear equations. The next chapter, on resultants and discriminants, contains a proof that the dialytic eliminant is the resultant, a treatment of subresultants and common roots of two equations, and the construction of discriminants in determinant form and in terms of the roots. The seventh chapter gives the solution, in much completeness, of the general equations of the second, third, and fourth orders, following in the latter cases the method of Euler.

The proofs given in Chapter eight that every algebraical equation has a root are respectively the first proof of Gauss, given in his Helmstädt dissertation in 1799, and a proof published by Cauchy in 1821.

Symmetric and alternating functions are treated in Chapter nine, the roots of unity by number-theoretic considerations in Chapter ten. The eleventh contains the theory of cyclotomy, regular polygons, and the f-nomial periods. Cyclic equations and their solution by radicals, and reciprocal equations form the subjects of the twelfth chapter. The succeeding chapters then lead up to the proof of the Abel-Ruffini theorem, treating substitution groups and their functions, and the solvability of algebraic equations in general. There then follows a chapter on transformation, with definitions and illustrations of invariants and covariants and reduction of the general n-ary quadratic form to a sum of squares, and a final chapter on Sturm's theorem.

Netto's treatment is, as a whole, particularly commendable, containing many a deft touch marking the handiwork of a master of his craft.

The standard of typography is high. A few misprints may be noted however: On page 3, line 7, αn should read α_n; on page 30, line 6, for $y \pm 2kn$ read $y \pm 2k\pi$; on page 59, last line, read a_k for a_κ; on page 85 in line 19, $a_1d - a_1d$ should read $ad_1 - a_1d$. The proof of page 106 has not been well read. There occur three notations for the same function on this page, viz., $g(x), f'(x)$, and $f_2(x)$. The order of f is m at the top of the page, is changed to n at the middle and so used in two determinants, there being no comment on the change, and the order m is restored at the bottom of the page. In line 3 from the bottom $(1/n^{n-1})R(f_2, f_1)$ should be $(1/n^{n-2})R(f_2, f_1)$. Moreover it is not good usage, we believe, to begin a sentence with a mathematical symbol instead of a capitalized word, as is done in the theorem given at the top of this page. In line 11 of page 115 the last β in the line is wrong font. On page 120 in line 15, $2\sqrt{(a_1^2 - a_0a_1)/a_0^2}$ should be $2\sqrt{(a_1^2 - a_0a_2)/a_0^2}$. In line 3 from the bottom of page 122 read a_4 for a_6. In line 2 of page 123 read x_3x_4 for x_3x_6. The numbering of the formulas in the region of page 123 is confused. Equation (27) referred to in line 6 of this page does not occur in the chapter. This renders line 16 on page 125 unintelligible although it may be a misprint of "Nun liefert (21) wegen (23)." On page 213 in line 4 read y_2 for y_2'.

In the way of general criticism the reviewer thinks it might be urged that the treatment of invariants in the book is much too brief. Quite probably this subject is to be expounded at greater length in the volumes on geometry. But if it could have been found feasible to introduce the notions of invariancy in connection with the solutions of the equations of orders less than 5, at sufficient length to show, for instance, that the roots of the resolvent cubic of the quartic equation are irrational invariants of index 2, the rôle of invariants in the elements of algebra would have been rendered more evident.

O. E. Glenn.

Solid Geometry. By William Betz, A.M., and Harrison E. Webb, A.B. With the editorial cooperation of Percy F. Smith. Ginn and Company, 1916. xxii+177 pp. Price $0.75.

On account of the existence of so many other interesting and important topics in mathematics which can be offered to the

man entering college there is a tendency to drop solid geometry from the curriculum altogether or to relegate it to the high schools. As the primary reason for the study of this subject we may assign the training in space perception and the additional knowledge of the universe which comes to the student; as a secondary, the additional training in logical thinking and expression. Since the former can be realized in a comparatively few lessons and the study of plane geometry *should* suffice for the latter, it is necessary that the colleges which offer this course to Freshmen be able to give an account of the faith which is in them.

It is true that the study of plane geometry is too often a mere memorizing of certain stock propositions rather than a training of the logical faculty, and that many freshmen show a lamentable ignorance of all methods of reasoning including those supposed to be geometrical; but the unbeliever will ask the pertinent question as to whether a really scientific course can be appreciated by the man entering college or whether it is better to postpone further study of deductive geometry until the junior or senior year. Certainly the great majority of texts which flood the market are ill-adapted for a course in proper reasoning. A not over-critical examination of one of the most popular texts in solid geometry shows errors in the statements or proofs of more than one third the theorems. In the case referred to it is probably the result of ignorance; but the authors of a recent text confess that the mass of errors introduced into their treatise is the result of deliberate catering to the infant mind. Unless considerable moral self-restraint is exercised in teaching, the use of such texts with a college class is apt to prejudice the student against mathematics if not against the instructor.

It ought to be possible in America, as it is in some other countries, to put geometry, both plane and solid, into interesting and understandable form without sacrificing logic. For solid geometry, Betz and Webb have done this at least as successfully as in any book in English that has come to the reviewer's attention. It is one of three or four texts which seem possible as a basis for a college course and it seems admirably adapted for high school use. Among the excellent features of the book are: a brief preliminary intuitional introduction to three-dimensional thinking; the grouping together of the fundamental axioms and some of the more intuitionally

obvious propositions as preliminary statements on which to base later demonstrational work; as adequate a treatment of the incommensurable as one could expect in an elementary course; sketches of proofs (such as that in regard to the volume of a cylinder) which might be made rigorous were the proper tools and the time available; a brief introduction to coordinates in space; a careful selection of propositions and exercises, an interesting style, and attractive typography.

While the great majority of the errors that are the bane of the ordinary text in solid geometry are here absent, some have been carried over to furnish targets for the critical mathematician. It may be well to cite here one each of three or four types. (1) In No. 592 concerning polyhedrons it is stated that "the lines of intersection of the bounding planes are called the edges; the points of intersection of the edges, the vertices." However, there may be many lines of intersection of the planes which are not edges and many points of intersection of the edges which are not vertices. (2) In proving (No. 555) that, if one of two parallel lines is perpendicular to a plane, the other is also, it is necessary first to prove that the second line meets the plane. (3) The notion of half-plane must be introduced into the discussion of diedral angles. Planes *will* extend beyond a line (see No. 559) whether we wish them to or not. The treatment of No. 579 needs an entire revision; among other criticisms it may be noted that the distance from a point to a half-plane face of a diedral angle has not been defined and can not be defined in any usable manner. In place of this theorem it would be better to introduce V of page 372 and the notions connected therewith. (4) In proving the theorem in regard to the volume of a triangular prism (No. 695) the bases of the two prisms are made to coincide. The question as to whether the prisms will then be on opposite sides of the coinciding bases or on the same side is one of order. To avoid this dilemma a mid-section parallel to the bases might be introduced. While no adequate treatment of the notion "order" is possible in an elementary text, its discussion in such a problem as this and of the orders of the face and diedral angles of two vertical polyedral angles (No. 811) warrants a much more careful and extended treatment. And are not the words "same order" in No. 870 used in an entirely different sense from that implied in No. 810?

Professor Smith has rendered a distinct service to the

mathematics of the country by his editing of several texts; this new volume should share in the wide recognition of worth accorded the series.

R. G. D. RICHARDSON.

The Calculus. By E. W. DAVIS and W. C. BRENKE. Edited by E. R. HEDRICK. New York, The Macmillan Company, 1913. xx+383+63 pp.

"THIS book attempts to preserve the essential features of the calculus, to give the student a thorough training in mathematical reasoning, to create in him a sure mathematical imagination, and to meet fairly the reasonable demand for enlivening and enriching the subject through applications at the expense of purely formal work that contains no essential principle."

This is the closing sentence of the preface. It sets forth four things that the authors attempted to do in writing the book. Probably every author of a calculus consciously attempts the first two. An examination of the current texts however reveals but little evidence that the last two have received adequate attention, although there is a clearly defined tendency towards a fuller recognition of their importance. While the formal type of calculus is pretty definitely standardized, there is no generally recognized norm for one of the type here under review. Accordingly a book of this kind is more difficult to write, and also more difficult to teach, than one of the former kind.

It is obvious to any one at all familiar with teachers of college mathematics that the genus is made up of two clearly defined species; namely, those who reverence the symbol and those whose main interest is in the thing symbolized. This book is obviously and confessedly not for the former. It makes its appeal to those who want our students of calculus to realize that the subject is not primarily a formal one, but that it is vitally connected with physical phenomena and represents an important and significant intellectual achievement of the race. For example, instead of devoting a large amount of space to a discussion of the artifices for integration, the authors have presented integration as a process of reversal of rates. They have done this admirably and have brought home to the student with clearness and force what the process is and why it is important for him to study it. And that is the

one thing needful. It helps to create in him a sure mathematical imagination—a far more important thing than to make him an expert manipulator of forms. It is impossible to do both in the time given to the subject.

There are many other features of the book that are excellent for the same reason. Prominent among these the presentation of Taylor's theorem may be mentioned. This presentation is unusual and has the great and rare merit of making the student see an important use for the theorem. The excellence of the treatment of relative rate of increase and the compound interest law is equally conspicuous. The treatment of simple and damped harmonic motion, the frequent and happy use of parametric equations, and the simple examples in least squares also merit special mention. There is a brief but adequate treatment of simple differential equations.

The book is not free from faults, although many of these are of a mechanical nature and are not inherent in the text. For example, the page is too condensed; a number of important principles and formulas appear in inconspicuous places and are not properly emphasized; it is difficult to use the book for reference. But not all of the shortcomings are of this nature. There are a number of inaccuracies. Some of the exercises are unsuitable. The treatment of infinite series is inadequate.

To sum up, it seems to the reviewer that the authors have been reasonably successful in their effort to enliven and enrich the subject and to present it in such a way as to help create in the student a sure mathematical imagination. He has used the book in his classes since its first appearance and imagines that his students get from it a clearer notion of the essential features of the calculus than they get from any other text he has used. Many of its faults can be overcome by the teacher, whereas its excellent features cannot so successfully be grafted by him upon a text of the formal type. In the reviewer's opinion it is the best elementary calculus now available for use in American colleges. A judicious combination of the good points of this book, the Osgood, and the Franklin, McNutt and Charles would make an ideal text.

WILLIAM BENJAMIN FITE.

Analytische Geometrie der Ebene. By PAUL CRANTZ. Leipzig, Teubner, 1915. v+93 pp.

THIS little book, labeled "Analytic Geometry," deals exclusively with the straight line and the conic sections and contains no discussion of the general second degree equation. As indicated on the title page, it is intended for self-instruction. It is arranged in a very compact manner and probably contains a maximum of material for a book of its size. The idea of a translation of axes is introduced early and is used consistently in getting the equations of the conic sections with axes parallel to the coordinate axes.

No clear-cut definition of the equation of a locus is given. In developing the equation of a locus the author simply shows that all points on the locus satisfy a certain equation, omitting the converse theorem. The methods used in getting the equation of a straight line seem very awkward and make what is really a very simple theorem appear rather complicated. In introducing each of the conic sections, a method of construction is given. This, as well as the discussion of the locus from its equation, is helpful to the self-teaching student.

R. B. ROBBINS.

Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik und Lebensversicherung. Von EMANUEL CZUBER. Erster Band, dritte Auflage. Leipzig, Teubner, 1914. xii+462 pages.

THE first volume of the second edition of this work appeared in 1908. It was reviewed by H. B. Phillips in this BULLETIN, volume 20, pages 429–431. In this notice of the first volume of the third edition it will therefore be sufficient to call attention to the nature of the changes made in producing it from the corresponding volume of the second edition.

In the six years intervening between the publication of these two editions of volume I the literature of the calculus of probabilities has been enriched by the appearance of several works of importance. Of these mention may be made of the following: Borel's suggestive and illuminating "Eléments de la Théorie des Probabilités"; Liebmann's German translation of Markoff's treatise in the Russian language; a new edition of Poincaré's lectures on the theory of probability revised by the author himself; the first volume of Bachelier's "Calcul des Probabilités"; Carvallo's "Calcul des Probabilités."

Naturally these works have brought out some new results and have emphasized some fresh points of view, at least in particular parts of the theory. But Czuber has not found it necessary to recast his treatise in order to take account of these. It retains the same general form and arrangement as heretofore. But in many places there are minor alterations and improvements and occasionally a new portion of several pages. In addition there are a few minor rearrangements of old matter. On the whole the work is considerably improved. The printing is well done, no typographical errors of importance having been found. On account of its great importance in the theory of probabilities one desires a more satisfactory account of Stirling's formula for the asymptotic character of the gamma function than that given in § 14.

The following is a list of the principal additions: an elegant section (pages 72–80) on the theory of mean value and various applications of it throughout the book; a discussion (pages 83–89) of the use of continuous variables in the theory of probabilities; derivation of formulas (pages 119–128) for the product of binomial factors; a section (pages 239–249) on "Spielprobleme"; an important chapter (pages 273–286) on continuous probabilities in which are developed the fundamental ideas about continuous probabilities in the sense of Bachelier's use of this term; additional matter (see especially pages 413–423) containing a selection of typical problems illustrating the applications in this direction of the analytical representation of arbitrary distributions.

Besides these larger sections, which may be singled out as distinct additions, there are many of less extent scattered throughout the whole volume and contributing essentially to its improvement. As an example of these one may mention the theorems of Bernoulli and Poisson which are now treated from various points of view and illuminated by various analytical lemmas. Through these several improvements the author has accomplished his purpose "den Inhalt nach manchen Richtungen zu erweitern und zu vertiefern."

R. D. CARMICHAEL.

Annuaire pour l'An 1916 *publié par le Bureau des Longitudes.* Paris, Gauthier-Villars, vi+502 pp., with two appendices.

AN excellent "Notice" by M. G. Bigourdan on the mean barometric pressure and law of the winds in France is the chief

feature of the *Annuaire* for 1916. This title is amply fulfilled in the text of the article with its full explanations and in the numerous diagrams. As with so many French writers, the author does not hesitate to start with a brief but clear explanation of the fundamental principles of meteorology, gradually carrying the reader towards the special line of thought which he is investigating, so that the whole constitutes an elementary treatise on the subject. There is also a sympathetic account of Commandant Guyou by M. Emile Picard. M. Guyou had long been connected with the naval activities of France and more particularly with various problems of navigation. He had taken an active part in the publication of the *Annuaire.*

The body of the publication is kept up to date in the usual manner. A useful novelty, and the only one noticed, is a supplement containing the chief astronomical events for the year 1917 compressed into eighteen pages.

ERNEST W. BROWN.

NOTES.

AT the meeting of the London mathematical society held on April 27 the following papers were read: By H. S. CARSLAW: "The Green's function for the equation $\nabla u^2 + k^2 u = 0$, II"; by S. CHAPMAN: "On the uniformity of gaseous density, according to the kinetic theory"; by J. HODGKINSON: "The nodal points of a plane sextic"; by P. A. MACMAHON: "Some problems of combinatory analysis"; by S. POLLARD: "On the deduction of criteria for the convergence of Fourier's series from Fejér's theorem concerning their summability"; by Mrs. G. C. YOUNG: "On the derivatives of a function"; by W. H. YOUNG: "Note on functions of upper and lower type."

AT the meeting of the Edinburgh mathematical society on May 12 the following papers were read: By S. BRODETSKY: "The linear differential equation of the second order"; by D. M. Y. SOMMERVILLE: "A new nomogram for the cubic equation"; by G. PHILIP: "On a group of parabolas associated with the triangle"; by F. G. TAYLOR: "Birationally related cubics."

THE National bureau of the census has recently published a bulletin on the United States life tables prepared by Professor

J. W. GLOVER, of the University of Michigan. Copies can be obtained gratis from the Bureau.

THE final date for receiving memoirs in competition for the King Gustav V prize (see BULLETIN for October, 1913, page 40) has been extended to October 31, 1917.

A new mathematical periodical, the *Revista de Matematicas*, has been founded at Buenos Aires. It will appear monthly under the editorship of Professor MANUEL GUITARTE.

THE following advanced courses in mathematics are announced for the academic year 1916–1917:

MASSACHUSETTS INSTITUTE OF TECHNOLOGY (first half-year):—By Professor H. W. TYLER: Theory of functions of a complex variable, two hours.—By Professor F. S. WOODS: Advanced calculus and differential equations, four hours. —By Professor F. H. BAILEY: Fourier series, two hours.—By Professor E. B. WILSON: Rigid dynamics and hydrodynamics, two hours.—By Professor C. L. E. MOORE: Analytic mechanics, two hours.—By Dr. J. LIPKA: Mathematical laboratory, two hours.

YALE UNIVERSITY:—By Professor JAMES PIERPONT: Theory of functions of a complex variable, two hours; Modern analytic geometry, two hours; Elliptic functions, two hours.—By Professor E. W. BROWN: Advanced calculus, three hours; Advanced dynamics, two hours; Celestial mechanics, two hours.—By Professor P. F. SMITH: Differential equations, two hours.—By ——————: Advanced algebra, two hours.—By Professor E. J. MILES and Dr. H. F. MACNEISH: Differential geometry, two hours.

MR. J. H. HILL has been appointed professor of mathematics at the Ohio Northern University.

PROFESSOR ELLEN HAYES, of Wellesley College, will retire from active service at the end of the present academic year.

MR. C. H. CURRIER, instructor in mathematics in Brown University, has been promoted to an assistant professorship.

Mr. F. S. Nowlan, of Columbia University, has been appointed instructor in mathematics at the Carnegie School of Technology at Pittsburgh.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Blutel (E.). Leçons de mathématiques spéciales à l'usage des candidats à l'Ecole Polytechnique et à l'Ecole Normale Supérieure, etc. Tome 1: Algèbre, ligne droite et plan, trigonométrie, analyse, applications géométriques. Tome 2: Géométrie analytique des courbes et des surfaces. Paris, Hachette, 1914. 8vo. 7+635+4+430 pp.

Bradgon (C.). Four dimensional vistas. New York, A. A. Knopl, 1916. $1.25

Chisini (O.). See Enriques (F.).

Dumont (E.). Théorie générale des nombres. Définitions fondamentales. (Collection Scientia.) Paris, Gauthier-Villars, 1915. 8vo. 194 pp. Fr. 2.00

Enriques (F.). Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche, pubblicate per cura dell' O. Chisini. Bologna, Zanichelli, 1915. 8vo. 14+397 pp. L. 12.00

Galdeano (Z. G. de). Razonamiento de mi curso elemental de calcolo infinitesimal comprendiendo nociones de matemática fisico-quimica. Zaragoza, G. Casanal, 1915. 8vo. 244 pp. 5 Pes.

Gonseth (F.). Etude synthétique et applications de l'apolarité. (Thèse, Zürich.) Genève, A. Kundig, 1915. 8vo. 60 pp.

Grönfeldt (S.). Systematisk förteckning öfver G. Mittag-Lefflers matematiska bibliotek. Upsala, 1915. 4to. 6+345 pp. M. 31.50

Hadamard (J.). Four lectures on mathematics delivered at Columbia University in 1911. (Ernest Kempton Adams Fund Publication No. 5.) New York, Columbia University Press, 1915. 4to. 6+52 pp. $0.75

Kuppis (J.). Die Beweisführung des grossen Satzes Fermats. Budapest, 1915. 8vo. 15 pp. M. 0.50

Loria (G.). Guida allo studio della storia delle matematiche. (Manuali Hoepli.) Milano, Hoepli, 1916. 16+228 pp. 12mo. L. 3.00

Sibirani (F.). Riassunto-formulario di geometria analitica, algebra, calcolo infinitesimale, calcolo vettoriale e meccanica razionale. Roma, tip. Nazionale, di G. Bertero, 1915. 8vo. 64 pp.

Størmer (C.). Den tredje Skandinaviske Matematikerkongress i Kristiania, 1913. Kristiania, H. Aschehoug, 1915. 8vo. 175 pp.

Teixeira (F. G.). Sur les problèmes célèbres de la géométrie élémentaire non résolubles avec la règle et le compas. Coïmbre, Imprimérie de l'Université, 1915. 4to. 132 pp.

Wahlund (A.). Sur quelques propriétés des fonctions entières de genre zéro. Upsala, 1915. 8vo. 75 pp. M. 3.00

II. ELEMENTARY MATHEMATICS.

ALVORD (C. P.) and DAVIS (M. E.). Drill and problem book in arithmetic. Syracuse, N. Y., Iroquois Publishing Company, 1916. 238 pp. $0.45

DAVIS (M. E.). See ALVORD (C. P.).

HUNT (B.). Community arithmetic. New York, American Book Company, 1916. 285 pp. 12mo. $0.60

KARVE (R. D.). Practical geometry. Part 1, for Indian schools. Part 2, for Indian colleges. London, Bell, 1915. Cr. 8vo. 1s.+1s. 6d.

MATHEMATICAL PAPERS for admission to the Royal Military College for the years 1906–15. London, Macmillan, 1916. 6s.

PETERS (J.). Dreistellige Tafeln für logarithmisches und numerisches Rechnen. Berlin, s. d. 8vo. 36 pp. Kartonniert. M. 1.00

REED (H. L.). Plane trigonometry. London, Bell, 1915. Cr. 8vo. 320 pp. 3s. 6d.

SHEPPARD (G. Q.). Final year's work in arithmetic. 2d edition. Pottstown, Pa., Hill School, 1915. 159 pp. $1.00

URAGUCHI (Y.). Handy logarithm tables. Tokyo, Uraguchi, 1915. 7 pp. $0.06

III. APPLIED MATHEMATICS.

BARKHAUSEN (H.). See MAXWELL (J. C.).

BARNES (A. A.). Hydraulic flow reviewed. London, Spon, 1916. 12s. 6d.

BARZIZZA (G. B.). Gnomica: l'orologia solare a tempo vero nella sua moderna applicazione. Milano, Hoepli, 1916. 8+199 pp. L. 2.50

BEAUFILS (H.). See HARANG (F.).

BERGERSEN (H.). See GRAN (A.).

BESTELMEYER (A.). See HANDBUCH.

BOCCARDI (G.). Lezioni di cosmografia. Milano, Hoepli, 1916. 9+233 pp. L. 3.00

BOSS (L.). Preliminary catalogue of 6188 stars for the epoch 1900, including those visible to the naked eye and other well-determined stars. Photo-reprint. Washington, 1915. 4to. 37+345 pp. $4.25

COLOMBI (C.). Potentiels et représentations géométriques de la thermodynamique. Paris, Dunod et Pinat, 1914. 95 pp. Fr. 4.50

DAVAUX (E.). See PERRY (J.).

DECKARD (H. C.). Handbook of practical mathematics. Birmingham, Mich., 1916. 82 pp. 16mo. $2.00

DRURY (F. E.). Geometry of building construction. 2d year course. London, Routledge, 1915. 238 pp. 3s.

EFFEMERIDI astronomiche ad uso dei naviganti per l'anno 1916. Genova, tip. r. Istituto idrografico, 1915. 8vo. 196 pp. L. 1.50

EMDE (F.). See MAXWELL (J. C.).

FABRY (E.). Problèmes de mécanique rationnelle à l'usage des candidats aux certificats de licence et à l'aggrégation. Paris, Hermann, 1915. 8vo. 425 pp. Fr. 12.00

FREUCHEN (P. B.). Termodynamik. Grundstraek af Termodynamikens Historie by de to Hovedsaetningers Betydning. Kjöbenhavn, 1915. 8vo. 152 pp. M. 3.60

GABRIEL (E.). Eléments de topographie et tracé des voies de communication. Paris, Gigord, 1915. 8vo. 652 pp.

GEHRCKE (E.). See HANDBUCH.

GOUARD (E.) et HIERNAUX (G.). Cours élémentaire de mécanique industrielle. Principes généraux, applications, exercices pratiques. Tome 1. 2e édition. Paris, Dunod et Pinat, 1914. 12mo. 386 pp. Cartonné. Fr. 4.50

GRAN (A.) og BERGERSEN (H.). Laerebok i navigation. 3. utgave' omarbeidet av O. Johnsen. Del 1. Kristiania, 1915. Gr. 8vo. 8+408 pp. Bound. M. 11.50

GRIESBACH (H.). Physikalisch-chemische Propädeutik. Unter besonderer Berücksichtigung der medizinischen Wissenschaften und der historischen und biographischen Angaben. 2te Hälfte. 4te Lieferung: Der Schall als besondere Form der mechanischen Energie; Register. Leipzig, 1915. Gr. 8vo. 37+353–1881 pp. (von Band II). M. 80.00

HALLWACHE (W.). See HANDBUCH.

HANDBUCH der Radiologie. Unter Mitwirkung von A. Bestelmeyer, O. Langevin, H. A. Lorentz, E. Rutherford, herausgegeben von E. Marx (4 Bände). Band 3: Glimmentladung, positive Säule und Lichtelektrizität, von E. Gehrcke, R. Seeliger und W. Hallwache. Leipzig, 1916. Gr. 8vo. 22+618 pp. M. 34.00

HARANG (F.) et BEAUFILS (H.). Notions élémentaires de géométrie descriptive appliquée au dessin. Paris, Dunod et Pinat, 1914. Cartonné. Fr. 2.50

HELDERMAN (W. D.). Thermodynamica van het Weston-Normaalelement. Utrecht, 1915. Gr. 8vo. 10+73 pp. M. 4.00

HENNING (F.). Die Grundlagen, Methoden und Ergebnisse der Temperaturmessung. Braunschweig, 1915. Gr. 8vo. 9+297 pp. M. 9.00

HIERNAUX (G.). See GOUARD (E.).

HIGBIE (H. H.). See TIMBIE (W. H.).

IZART (J.). Agenda mécanique à l'usage des ingénieurs, constructeurs-mécaniciens, industriels, chefs d'ateliers et contremaîtres. Paris, Dunod et Pinat, 1916. Toile souple. Fr. 3.00

JEANS (J. H.). The dynamical theory of gases. 2d edition. Cambridge, University Press, 1916. 6+436 pp. 16s.

JOHNSEN (O.). See GRAN (A.).

KOPPE (M.). Die Bahnen der beweglichen Gestirne im Jahre 1916. Berlin, 1916. Gr. 8vo. 10 pp. M. 0.40

LAMB (H.). Hydrodynamics. 4th edition. Cambridge, University Press, 1916. 16+708 pp. 24s.

LANGEVIN (O.). See HANDBUCH.

LORENTZ (H. A.). See HANDBUCH.

MAILLET (E.). Cours de mécanique. Paris, Hermann, 1915. 8vo. 376 pp. Fr. 10.00

MARTENSEN-LARSEN (H.). Stjernehimlens store Problemer. Kjöbenhavn, 1915. 8vo. 330 pp. M. 6.40

MARTIN (L. A.). Text-book of mechanics. Vol. VI: Thermodynamics. New York, Wiley, 1916. 18+313 pp. Cloth. $1.75

MARX (E.). See HANDBUCH.

MATZDORFF (M.). Berechnung des Mondradius aus Bedeckungen von Sternen erster Grösse während der Jahre 1831–1911. Strassburg, 1914. 4to. 60 pp. M. 2.00

MAXWELL (J. C.). Elektrizität und Magnetismus. Auszüge. Uebersetzt von H. Barkhausen, herausgegeben von F. Emde. Braunschweig, 1915. Gr. 8vo. 32+182 pp. M. 7.00

PARKER (G. W.). Elements of optics for the use of schools and colleges. London and New York, Longmans, 1915. 8vo. 6+122 pp. Cloth. $0.75

PERREGAUX (C.) et WEBER (A.). Le relief en géométrie par les couleurs complémentaires. 50 planches de géométrie et de géométrie descriptive. Bienne, E. Magnon, 1915. Fr. 25.00

PERRY (J.). Mécanique appliquée à l'usage des élevès qui peuvent travailler expérimentalement et faire des exercices numériques et graphiques. Ouvrage traduit sur la 9e édition anglaise par E. Davaux. Avec un appendice sur les "toupies tournantes" du même auteur. Tome 2: Constructions déformables et machines en mouvement. Paris, Hermann, 1915. 8vo. 319 pp. Fr. 8.00

PLANCK (M.). Eight lectures on theoretical physics delivered at Columbia University in 1909. Translated by A. P. Wills. (Ernest Kempton Adams Fund Publication No. 3.) New York, Columbia University Press, 1915. 4to. 10+130 pp. $1.00

ROGERS (G. A.). Handbook of fundamental optics. Kansas City, 1915. 8vo. 98 pp. $1.75

RUTHERFORD (E.). See HANDBUCH.

SCHAU (A.). Statik mit Einschluss der Festigkeitslehre. (Aus Natur und Geisteswelt, No. 497.) Leipzig, Teubner, 1915. 8vo. 4+144 pp. M. 1.25

SEAL (B.). The positive sciences of the ancient Hindus. London, Longmans, 1915. 8+295 pp. 12s. 6d.

SEELIGER (R.). See HANDBUCH.

TIMBIE (W. H.) and HIGBIE (H. H.). Alternating-current electricity and its applications to industry. Second course. (Wiley Technical Series.) New York, Wiley, 1916. 8vo. 10+729 pp. Cloth. $3.00

WALKER (M.). Specification and design of dynamo-electric machinery. New York, 1915. 4to. 19+648 pp. $12.00

WEBER (A.). See PERREGAUX (C.).

WELD (L. D.). Theory of errors and least squares. New York, Macmillan, 1916. 12mo. 190 pp. $1.25

WERELDE (T.). Statical theory of energy and matter. Copenhagen, 1915. 8vo. 156 pp. M. 8.50

WILLS (A. P.). See PLANCK (M.).

WILMOTTE (M.). Cours de mécanique à l'usage des écoles industrielles et professionelles. Paris, Béranger, 1915. Cartonné. Fr. 10.00

TWENTY-FIFTH ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

ALEXANDER, J. W., II. Functions which map the interior of the unit circle upon simple regions. Read April 24, 1915. *Annals of Mathematics*, ser. 2, vol. 17, No. 1, pp. 12–22; Sept., 1915.

ALTSHILLER, N. On the circles of Apollonius. Read Aug. 3, 1915. *American Mathematical Monthly*, vol. 22, No. 8, pp. 261–263; Oct., 1915: No. 9, pp. 304–305; Nov., 1915.

BARROW, D. F. Oriented circles in space. Read Jan. 2, 1915. *Transactions of the American Mathematical Society*, vol. 16, No. 3, pp. 235–258; July, 1915.

BATEMAN, H. A certain system of linear partial differential equations. Read Feb. 26, 1916. *Bulletin of the American Mathematical Society*, vol. 22, No. 7, pp. 329–335; April, 1916.

BEETLE, R. D. On the complete independence of Schimmack's postulates for the arithmetic mean. Read April 25, 1914. *Mathematische Annalen*, vol. 76, No. 4, pp. 444–446; June, 1915.

—— Congruences associated with a one-parameter family of curves. Read Dec. 31, 1913 and Sept. 8, 1914. *American Journal of Mathematics*, vol. 37, No. 3, pp. 281–308; July, 1915.

BENNETT, A. A. The iteration of functions of one variable. Read Aug. 4, 1915. *Annals of Mathematics*, ser. 2, vol. 17, No. 1, pp. 23–60; Sept. 1915.

BERNSTEIN, B. A. A set of four independent postulates for Boolean algebras. Read Aug. 3, 1915. *Transactions of the American Mathematical Society*, vol. 17, No. 1, pp. 50–52; Jan., 1916.

—— A simplification of the Whitehead-Huntington set of postulates for Boolean algebras. Read (San Francisco) Nov. 20, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 9, pp. 458–459; June, 1916.

BIRKHOFF, G. D. The restricted problem of three bodies. Read Dec. 30, 1913 and April 25, 1914. *Rendiconti del Circolo Matematico di Palermo*, vol. 39, No. 3, pp. 265–334; May–June, 1915.

—— Theorem concerning the singular points of ordinary linear differential equations. Read Oct. 30, 1915. *Proceedings of the National Academy of Sciences*, vol. 1, No. 12, pp. 578–581; Dec., 1915.

BLISS, G. A. A note on the problem of Lagrange in the calculus of variations. Read Dec. 31, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 5, pp. 220–225; Feb., 1916.

—— Jacobi's condition for problems of the calculus of variations in parametric form. Read (Chicago) April 2, 1915. *Transactions of the American Mathematical Society*, vol. 17, No. 2, pp. 195–206; April, 1916.

BLUMBERG, H. On the factorization of various types of expressions. Read (Southwestern Section) Nov. 28, 1914 and (Chicago) Dec. 29, 1914. *Proceedings of the National Academy of Sciences*, vol. 1, No. 6, pp. 374–381; June, 1915.

BUCHANAN, D. Oscillations near one of the isosceles-triangle solutions of the three body problem. Read Jan. 1, 1913. *Proceedings of the London Mathematical Society*, ser. 2, vol. 14, No. 4, pp. 278–300; July, 1915.

—— A new isosceles-triangle solution of the three body problem. Read (Chicago) Dec. 28, 1914. *Transactions of the American Mathematical Society*, vol. 16, No. 3, pp. 259–274; July, 1915.

CAJORI, F. The history of Zeno's arguments on motion: phases in the development of the theory of limits. Read Sept. 9, 1913 and (Chicago) April 11, 1914. *American Mathematical Monthly*, vol. 22, No. 5, pp. 143–149; May, 1915: No. 6, pp. 179–186; June, 1915: No. 7, pp. 215–220; Sept., 1915: No. 8, pp. 253–258; Oct., 1915: No. 9, pp. 292–297; Nov., 1915.

CARMICHAEL, R. D. On the representation of numbers in the form $x^3+y^3+z^3-3xyz$. Read Aug. 3, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 3, pp. 111–117; Dec., 1915.

—— Diophantine analysis (Mathematical Monograph Series, No. 16); pp. 35–58, 62–66, 77–84, 104–111. Read (Chicago), April 3, 1915. New York, Wiley, 1915.

—— On the solutions of linear homogeneous difference equations. Read (Chicago) April 3, 1915. *American Journal of Mathematics*, vol. 38, No. 2, pp. 185–220; April, 1916.

CARPENTER, A. F. Ruled surfaces whose flecnode curves have plane branches. Read (Chicago) Dec. 28, 1914. *Transactions of the American Mathematical Society*, vol. 16, No. 4, pp. 509–532; Oct., 1915.

COBLE, A. B. An isomorphism between theta characteristics and the $(2p+2)$-point. Read Jan. 1, 1915. *Annals of Mathematics*, ser. 2, vol. 17, No. 3, pp. 101–112; March, 1916.

CUMMINGS, L. D., and WHITE, H. S. Groupless triad systems on fifteen elements. Read April 24, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 1, pp. 12–16; Oct., 1915.

CURTISS, D. R. Extensions of Descartes' rule of signs connected with a problem suggested by Laguerre. Read Jan. 1, 1913 and (Chicago) Dec. 26, 1913. *Transactions of the American Mathematical Society*, vol. 16, No. 3, pp. 350–360; July, 1915.

DANIELL, P. J. The coefficient of end-correction. Read (Southwestern Section) Nov. 28, 1914. *Philosophical Magazine*, ser. 6, vol. 30, No. 175, pp. 137–146; July, 1915: No. 176, pp. 248–256; Aug., 1915.

DICKSON, L. E. Geometrical and invariantive theory of quartic curves modulo 2. Read (Chicago) April 2, 1915. *American Journal of Mathematics*, vol. 37, No. 4, pp. 337–354; Oct., 1915.

—— Invariantive classification of pairs of conics modulo 2. Read Aug. 3, 1915. *American Journal of Mathematics*, vol. 37, No. 4, pp. 355–358; Oct., 1915.

—— On the twenty-eight bitangents to a quartic curve. Read (Chicago) April 11, 1914. Chapter 19 of Theory and applications of finite groups, by G. A. Miller, H. F. Blichfeldt and L. E. Dickson. New York, Wiley, 1916.

EISENHART, L. P. Conjugate systems with equal tangential invariants and the transformation of Moutard. Read Sept. 8, 1914. *Rendiconti del Circolo Matematico di Palermo*, vol. 39, No. 2, pp. 153–176; March–April, 1915.

—— Surfaces Ω and their transformations. Read Jan. 1, 1915. *Transactions of the American Mathematical Society*, vol. 16, No. 3, pp. 275–310; July, 1915.

—— Surfaces with isothermal representation of their lines of curvature as envelopes of rolling. Read Oct. 30, 1915. *Annals of Mathematics*, vol. 17, No. 2, pp. 64–71; Dec., 1915.

—— Transformations of surfaces Ω (second memoir). Read Dec. 28, 1915. *Transactions of the American Mathematical Society*, vol. 17, No. 1, pp. 53–99; Jan., 1916.

EMCH, A. A certain class of functions connected with Fuchsian groups. Read April 24, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 1, pp. 33–37; Oct., 1915.

EPPERSON, C. A. Note on Green's theorem. Read April 24, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 1, pp. 17–26; Oct., 1915.

EVANS, G. C. The non-homogeneous differential equation of parabolic type. Read Sept. 8, 1914. *American Journal of Mathematics*, vol. 37, No. 4, pp. 431–438; Oct., 1915.

—— Application of an equation in variable differences to integral equations. Read April 29, 1916. *Bulletin of the American Mathematical Society*, vol. 22, No. 10, pp. 493–503; July, 1916.

FORD, W. B. On the representation of arbitrary functions by definite integrals. Read (Chicago) April 3, 1915. *Proceedings of the National Academy of Sciences*, vol. 1, No. 7, pp. 431–435; July, 1915.

FORSYTH, C. H. A general formula for the valuation of bonds. Read (Chicago) Dec. 29, 1914. *American Mathematical Monthly*, vol. 22, No. 5, pp. 149–152; May, 1915.

—— Osculatory interpolation formulas. Read Feb. 27, 1915. *Quarterly Publications of the American Statistical Association*, vol. 14, No. 110, pp. 583–589; June, 1915.

FRÉCHET, M. Sur les fonctionnelles bilinéaires. Read Feb. 27, 1915, *Transactions of the American Mathematical Society*, vol. 16, No. 3. pp. 215–234; July, 1915.

—— On Pierpont's definition of integrals. Read Dec. 27, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 6, pp. 295–298; March, 1916.

FRIZELL, A. B. The permutations of the natural numbers can not be well ordered. Read Feb. 27, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 2, pp. 71–73; Nov., 1915.

GILLESPIE, D. C. The Cauchy definition of a definite integral. Read Sept. 9, 1914. *Annals of Mathematics*, ser. 2, vol. 17, No. 2, pp. 61–63; Dec., 1915.

GREEN, G. M. Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface. Read Oct. 25, 1913. *American Journal of Mathematics*, vol. 37, No. 3, pp. 215–246; July, 1915.

—— On isothermally conjugate nets of space curves. Read Aug. 4, 1915. *Proceedings of the National Academy of Sciences*, vol. 1, No. 10, pp. 516–521; Oct., 1915.

—— On the linear dependence of functions of several variables, and certain completely integrable systems of partial differential equations. Read Oct. 30, 1915. *Proceedings of the National Academy of Sciences*, vol. 2, No. 4, pp. 209–214; April, 1916.

GRIFFIN, F. L. An experiment in correlating freshman mathematics. Read (San Francisco) May 22, 1914. *American Mathematical Monthly*, vol. 22, No. 10, pp. 325–330; Dec., 1915.

GRONWALL, T. H. A functional equation in the kinetic theory of gases. Read April 24, 1915. *Annals of Mathematics*, ser. 2, vol. 17, No. 1, pp. 1–4; Sept., 1915.

—— Determination of all triply orthogonal systems containing a family of minimal surfaces. Read April 25, 1914. *Annals of Mathematics*, ser. 2, vol. 17, No. 2, pp. 76–100; Dec., 1915.

—— Sur une équation fonctionnelle dans la théorie cinétique des gaz. Read Feb. 26, 1916. *Comptes Rendus de l'Académie des Sciences*, vol. 162, No. 12, pp. 415–418; March 20, 1916.

HASKINS, C. N. On the zeros of the function $P(x)$ complementary to the incomplete gamma function. Read Jan. 2, 1915. *Transactions of the American Mathematical Society*, vol. 16, No. 4, pp. 405–412; Oct., 1915.

—— On the measurable bounds and the distribution of functional values of summable functions. Read Oct. 30, 1915. *Transactions of the American Mathematical Society*, vol. 17, No. 2, pp. 181–194; April, 1916.

H'DOUBLER, F. T. See VAN VLECK, E. B.

HOSKINS, L. M. Mass as quantity of matter. Read Aug. 4, 1915. *Science*, new ser., vol. 42, No. 1080, pp. 340–341; Sept. 10, 1915.

—— "Quantity of matter" in dynamics. Read Aug. 4, 1915. *American Mathematical Monthly*, vol. 23, No. 2, pp. 34–41; Feb., 1916.

IRWIN, F. A curious convergent series. Read (San Francisco) Nov. 20, 1915. *American Mathematical Monthly*, vol. 23, No. 5, pp. 149–152; May, 1916.

JACKSON, D. A formula of trigonometric interpolation. Read Sept. 8, 1913. *Rendiconti del Circolo Matematico di Palermo*, vol. 39, No. 2, pp. 230–232; March–April, 1915.

—— Expansion problems with irregular boundary conditions. Read Jan. 1, 1915. *Proceedings of the American Academy of Arts and Sciences*, vol. 51, No. 7, pp. 381–417; Nov., 1915.

—— Note on rational functions of several complex variables. Read Feb. 28, 1914. *Journal für die reine und angewandte Mathematik*, vol. 146, No. 3, pp. 185–188; Jan., 1916.

—— Proof of a theorem of Haskins. Read Aug. 3, 1915. *Transactions of the American Mathematical Society*, vol. 17, No. 2, pp. 178–180; April, 1916.

—— An elementary boundary value problem. Read April 29, 1916. *Bulletin of the American Mathematical Society*, vol. 22, No. 8, pp. 393–397; May, 1916.

KASNER, E. Conformal classification of analytic arcs or elements: Poincaré's local problem of conformal geometry. Read Oct. 25, 1913. *Transactions of the American Mathematical Society*, vol. 16, No. 3, pp. 333–349; July, 1915.

—— Infinite groups generated by conformal transformations of period two (involutions and symmetries). Read Sept. 9, 1914 and Dec. 27, 1915. *American Journal of Mathematics*, vol. 38, No. 2, pp. 177–184; April, 1916.

KELLOGG, O. D. The oscillation of functions of an orthogonal set. Read (Southwestern Section) Nov. 29, 1913. *American Journal of Mathematics*, vol. 38, No. 1, pp. 1–5; Jan., 1916.

KEYSER, C. J. The human significance of mathematics. Read Aug. 3, 1915. *Science*, new ser., vol. 42, No. 1089, pp. 663–680; Nov. 12, 1915.

KIRCHER, E. Group properties of the residue classes of certain Kronecker modular systems and some related generalizations in number theory. Read (Chicago) April 10, 1914. *Transactions of the American Mathematical Society*, vol. 16, No. 4, pp. 413–434; Oct., 1915.

LAMOND, J. K. The reduction of multiple L-integrals of separated functions to iterated L-integrals. Read Dec. 31, 1913. *Transactions of the American Mathematical Society*, vol. 16, No. 4, pp. 387–398; Oct., 1915.

LEFSCHETZ, S. On cubic surfaces and their nodes. Read (San Francisco) April 6, 1912. *Kansas University Science Bulletin*, vol. 9, No. 6, pp. 69–78; Dec., 1914.

—— Note on the n-dimensional cycles of an algebraic n-dimensional variety. Read (Southwestern Section) Nov. 27, 1915. *Rendiconti del Circolo Matematico di Palermo*, vol. 40, No. 1, pp. 38–43; July–Aug., 1915.

LINEHAN, P. H. Contributions to equilong geometry. Read Feb. 27, 1915. Author's Dissertation. Lancaster, 1915. 6+38 pp.

LOVE, C. E. On linear difference and differential equations. Read (Chicago) April 2, 1915. *American Journal of Mathematics*, vol. 38, No. 1, pp. 57–80; Jan., 1916.

LOVITT, W. V. A type of singular points for a transformation of three variables. Read (Chicago) April 2, 1915. *Transactions of the American Mathematical Society*, vol. 16, No. 4, pp. 371–386; Oct., 1915.

—— A type of singular points for a transformation of three variables. Read Dec. 31, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 5, pp. 236–239; Feb., 1916.

MACMILLAN, W. D. Convergence of the series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^iy^j}{i-j\gamma}$ (γ irrational). Read (Chicago) April 3, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 1, pp. 26–32; Oct., 1915.

MASON, T. E. Mechanical device for testing Mersenne numbers for primes. Read Sept. 9, 1914. *Proceedings of the Indiana Academy of Science*, 1914, pp. 429–431.

—— On properties of the solutions of linear q-difference equations with entire function coefficients. Read (Chicago) April 11, 1914. *American Journal of Mathematics*, vol. 37, No. 4, pp. 439–444; Oct., 1915.

MILLER, B. I. A new canonical form of the elliptic integral. Read Dec. 27, 1915. *Proceedings of the National Academy of Sciences*, vol. 1, No. 5, pp. 274–275; May, 1915.

MILLER, G. A. Note on several theorems due to A. Capelli. Read Jan. 1, 1915. *Giornale di Matematiche*, vol. 53, Nos. 4–5, pp. 313–315; July–Oct., 1915.

—— Independent generators of a group of finite order. Read (Chicago) April 3, 1915. *Transactions of the American Mathematical Society*, vol. 16, No. 4, pp. 399–404; Oct., 1915.

—— Limits of the degree of transitivity of substitution groups. Read Aug. 3, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 2, pp. 68–71; Nov., 1915.

—— Upper limit of the degree of transitivity of a substitution group. Read Jan. 1, 1916. *Proceedings of the National Academy of Sciences*, vol. 2, No. 1, pp. 61–62; Jan., 1916.

MISER, W. L. On multiform solutions of linear differential equations having elliptic function coefficients. Read (Chicago) March 21, 1913. *Transactions of the American Mathematical Society*, vol. 17, No. 2, pp. 109–130; April, 1916.

MITCHELL, H. H. On the generalized Jacobi-Kummer cyclotomic function. Read Dec. 28, 1915. *Transactions of the American Mathematical Society*, vol. 17, No. 2, pp. 165–177; April, 1916.

MOORE, C. L. E. See WILSON, E. B.

MOORE, R. L. On the linear continuum. Read April 24, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 3, pp. 117–122; Dec., 1915.

—— Concerning a non-metrical pseudo-Archimedean axiom. Read April 26, 1913. *Bulletin of the American Mathematical Society*, vol. 22, No. 5, pp. 225–236; Feb., 1916.

—— On the foundations of plane analysis situs. Read April 24, 1915. *Transactions of the American Mathematical Society*, vol. 17, No. 2, pp. 131–164; April, 1916. *Proceedings of the National Academy of Sciences*, vol. 2, No. 5, pp. 270–272; May, 1916.

MOULTON, E. J. On figures of equilibrium of a rotating compressible fluid mass; certain negative results. Read (Chicago) Dec. 27, 1913. *Transactions of the American Mathematical Society*, vol. 17, No. 1, pp. 100–108; Jan., 1916.

NELSON, A. L. Quasi-periodicity of asymptotic plane nets. Read (Chicago) April 21, 1916. *Bulletin of the American Mathematical Society*, vol. 22, No. 9, pp. 445–455; June, 1916.

OSGOOD, W. F. On functions of several complex variables. Read Oct. 30, 1915. *Transactions of the American Mathematical Society*, vol. 17, No. 1, pp. 1–8; Jan., 1916.

—— Note on functions of several complex variables. Read April 29, 1916. *Bulletin of the American Mathematical Society*, vol. 22, No. 9, pp. 443–445; June, 1916.

PFEIFFER, G. A. On the conformal geometry of analytic arcs. Read Oct. 31, 1914. *American Journal of Mathematics*, vol. 37, No. 4, pp. 395–430; Oct., 1915.

POOR, V. C. Transformation theorems in the theory of the linear vector function. Read Dec. 31, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 4, pp. 174–181; Jan., 1916.

RANUM, A. Duality in the differential geometry of space-curves. Read Sept. 6, 1910. *Quarterly Journal of Pure and Applied Mathematics*, vol. 46, No. 4, pp. 356–384; Oct., 1915.

RICHARDSON, L. J. Digital reckoning among the ancients. Read Aug. 3, 1915. *American Mathematical Monthly*, vol. 23, No. 1, pp. 7–13; Jan., 1916.

RITT, J. F. On certain real solutions of Babbage's functional equation. Read Feb. 27, 1915. *Annals of Mathematics*, ser. 2, vol. 17, No. 3, pp. 113–122; March, 1916.

ROWE, J. E. Relations among parameters along the rational cubic curve. Read April 24, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 2, pp. 74–76; Nov., 1915.

—— A new method of finding the equation of a rational plane curve from its parametric equations. Read Dec. 27, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 7, pp. 338–340; April, 1916.

SAFFORD, F. H. An irrational transformation of the Weierstrass $\wp$-function curves. Read April 24, 1915. *Archiv der Mathematik und Physik*, ser. 3, vol. 24, No. 4, pp. 342–344; March, 1916.

SHAW, J. B. On parastrophic algebras. Read (Chicago) Dec. 29, 1914. *Transactions of the American Mathematical Society*, vol. 16, No. 3, pp. 361–370; July, 1915.

SISAM, C. H. On rational sextic surfaces having a nodal curve of order 9. Read (Chicago) April 2, 1915. *American Journal of Mathematics*, vol. 37, No. 4, pp. 445–456; Oct., 1915.

—— On surfaces doubly generated by conics. Read Dec. 31, 1915. *Quarterly Journal of Pure and Applied Mathematics*, vol. 47, No. 1, pp. 55–72; March, 1916.

—— On a configuration on certain surfaces. Read (Chicago) April 21, 1916. *Bulletin of the American Mathematical Society*, vol. 22, No. 8, pp. 381–383; May, 1916.

STEIMLEY, L. L. On the solutions of linear non-homogeneous partial differential equations. Read (Chicago) Dec. 28, 1914. *American Journal of Mathematics*, vol. 37, No. 4, pp. 359–366; Oct., 1915.

STOUFFER, E. B. On seminvariants of linear homogeneous differential equations. Read (Chicago) April 3, 1915. *Proceedings of the London Mathematical Society*, ser. 2, vol. 15, No. 3, pp. 217–226; May, 1916.

TAPPAN, A. H. Plane sextic curves invariant under birational transformations. Read Dec. 30, 1913. *American Journal of Mathematics*, vol. 37, No. 3, pp. 309–336; July, 1915.

VALLÉE POUSSIN, C. J. DE LA. Sur l'intégrale de Lebesgue. Read April 24, 1915 and Aug. 3, 1915. *Transactions of the American Mathematical Society*, vol. 16, No. 4, pp. 435–501; Oct., 1915.

VANDIVER, H. S. An aspect of the linear congruence with applications to the theory of Fermat's quotient. Read Aug. 4, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 2, pp. 61–67; Nov., 1915.

VAN VLECK, E. B. and H'DOUBLER, F. T. A study of certain functional equations for the ϑ-functions. Read Sept. 9, 1913. *Transactions of the American Mathematical Society*, vol. 17, No. 1, pp. 9–49; Jan., 1916.

WAHLIN, G. E. A new development of the theory of algebraic numbers. Read (Chicago) April 2, 1915. *Transactions of the American Mathematical Society*, vol. 16, No. 4, pp. 502–508; Oct., 1915.

WEDDERBURN, J. H. M. On matrices whose coefficients are functions of a single variable. Read Sept. 9, 1914. *Transactions of the American Mathematical Society*, vol. 16, No. 3, pp. 328–332; July, 1915.

WHITE, H. S. Seven points on a twisted cubic curve. Read Aug. 3, 1915. *Proceedings of the National Academy of Sciences*, vol. 1, No. 8, pp. 464–466; Aug., 1915.

—— Poncelet polygons. Read Dec. 30, 1915. *Science*, new ser., vol. 43, No. 1101, pp. 149–158; Feb. 4, 1916.

—— See CUMMINGS, L. D.

WILCZYNSKI, E. J. The general theory of congruences. Read (Chicago) Dec. 28, 1914. *Transactions of the American Mathematical Society*, vol. 16, No. 3, pp. 311–327; July, 1915.

—— Some remarks on the historical development and the future prospects of the differential geometry of plane curves. Read (Chicago) April 11, 1914 and Dec. 30, 1915. *Bulletin of the American Mathematical Society*, vol. 22, No. 7, pp. 317–329; April, 1916.

—— Interpretation of the simplest integral invariant of projective geometry. Read Jan. 1, 1916. *Proceedings of the National Academy of Sciences*, vol. 2, No. 4, pp. 248–252; April, 1916.

WILDER, C. E. On the degree of approximation to discontinuous functions by trigonometric sums. Read Feb. 22, 1913 and April 25, 1914. *Rendiconti del Circolo Matematico di Palermo*, vol. 39, No. 3, pp. 345–361; May–June, 1915.

WILLIAMS, K. P. A theorem concerning real functions. Read (Southwestern Section) Nov. 27, 1915. *Annals of Mathematics*, ser. 2, vol. 17, No. 2, pp. 72–73; Dec., 1915.

—— Concerning Hill's derivation of the Lagrange equations of motion. Read (Chicago) April 22, 1916. *Bulletin of the American Mathematical Society*, vol. 22, No. 9, pp. 455–457; June, 1916.

WILSON, E. B. Linear momentum, kinetic energy, and angular momentum. Read April 24, 1915. *American Mathematical Monthly*, vol. 22, No. 6, pp. 187–193; June, 1915.

—— Changing surface to volume integrals. Read Feb. 26, 1916. *Bulletin of the American Mathematical Society*, vol. 22, No. 7, pp. 336–337; April, 1916.

—— Critical speeds for flat disks in a normal wind. Prefatory note on normal flow past a circular disk. Read Feb. 26, 1916. *Smithsonian Miscellaneous Collections*, vol. 62, No. 4, pp. 77–83; Jan. 15, 1916.

WILSON, E. B. and MOORE, C. L. E. A general theory of surfaces. Read Dec. 27, 1915. *Proceedings of the National Academy of Sciences*, vol. 2, No. 5, pp. 273–278; May, 1916.

INDEX OF VOLUME XXII.

REVIEWS.

Notes and Other Items.

AMB Math

BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

A Historical and Critical Review of Mathematical Science

Volume XXII., Number 10

July, 1916

PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK
1916

Entered at the Post Office at Lancaster, Pa., as second class matter
Published Monthly, except August and September

Whole No. 250 $5.00 a year

American Mathematical Society

Colloquium Lectures

I. The Boston Colloquium Lectures on Mathematics

I. Linear Systems of Curves on Algebraic Surfaces. By H. S. WHITE.

II. Forms of Non-Euclidean Space. By F. S. WOODS.

III. Selected Topics in the Theory of Divergent Series and of Continued Fractions. By E. B. VAN VLECK.

Delivered before the Fourth Colloquium of the Society at the Massachusetts Institute of Technology, Boston, Mass., September 2–5, 1903.

Published for the Society, 1905, by The Macmillan Company, New York, N. Y. Orders should be addressed to the Society, 501 West 116th Street, New York City. Price, $2.00; to members of the Society, $1.50.

II. The New Haven Mathematical Colloquium

I. Introduction to a Form of General Analysis. By E. H. MOORE.

II. Projective Differential Geometry. By E. J. WILCZYNSKI.

III. Selected Topics in the Theory of Boundary Value Problems of Differential Equations. By MAX MASON.

Delivered before the Fifth Colloquium of the Society at Yale University, New Haven, Conn., September 3–8, 1906.

Published, 1910, by the Yale University Press, New Haven, Conn., to which orders should be addressed. Price, $3.00 ; postage, 17 cents.

American Mathematical Society

Colloquium Lectures

III. The Princeton Colloquium Lectures

I. Fundamental Existence Theorems. By G. A. Bliss.

II. Differential-Geometric Aspects of Dynamics. By Edward Kasner.

Delivered before the Sixth Colloquium of the Society at Princeton University, Princeton, N. J., September 15–17 1909.

Published, 1913, by the Society, to which orders should be addressed. Price, $1.50 ; to members of the Society, $1.00.

IV. The Madison Colloquium Lectures

I. On Invariants and the Theory of Numbers. By L. E. Dickson.

II. Topics in the Theory of Functions of Several Complex Variables. By W. F. Osgood.

Delivered before the Seventh Colloquium of the Society at the University of Wisconsin, Madison, Wis., September 10–13, 1913.

Published, 1914, by the Society, to which orders should be addressed, Price, $2.00 ; to members of the Society, $1.50.

Whole No. 250

CONTENTS

Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the American Mathematical Society, 501 West 116th Street, New York.

Subscriptions to the Bulletin, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to The American Mathematical Society, 41 North Queen St., Lancaster, Pa., or 501 West 116th Street, New York.

The initiation fees and annual dues of members of the American Mathematical Society are payable to the Treasurer of the American Mathematical Society, Professor J. H. Tanner, Cornell University, Ithaca, N. Y.

Articles for insertion in the Bulletin should be addressed to the

Bulletin of the American Mathematical Society,

501 West 116th Street, New York, N. Y.

The following dates have been fixed for the meetings of the Society:

	A. M.		A. M.
Sat., Oct. 28, 1916,	11:00	Sat., Feb. 24, 1917,	11:00
Wed.-Th., Dec. 27-28, 1916.		Sat., Apr. 28, "	"

The Twenty-first Summer Meeting and Eighth Colloquium of the Society will be held at Harvard University, September 4-9, 1916.

THE NEW ERA PRINT, LANCASTER, PA.